$ab + bc + bd =$

$-18 \cancel{- 6\sqrt{3}} - 18 \cancel{+ 6\sqrt{3}}$

$+9\sqrt{3}$

Property
of
Frank C. Frencik
1918

COMPLETE

SECONDARY ALGEBRA

QUADRATICS AND BEYOND

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COMPLETE

SECONDARY ALGEBRA

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QUADRATICS AND BEYOND

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CHAPTER XVIII.

QUADRATIC EQUATIONS.

1. A **Quadratic Equation** is an equation of the second degree in the unknown number or numbers.

E.g., $x^2 = 25$, $x^2 - 5x + 6 = 0$, $x^2 + 2xy = 7$.

A **Complete Quadratic Equation,** in one unknown number, is one which contains a term (or terms) in x^2, a term (or terms) in x, and a term (or terms) free from x, as $x^2 - 2ax + b = cx - d$.

A **Pure Quadratic Equation** is an incomplete quadratic equation which has no term in x, as $x^2 - 9 = 0$.

Pure Quadratic Equations.

2. Ex. 1. Solve the equation $6x^2 - 7 = 3x^2 + 5$.

Transferring $3x^2$ to the first member, and 7 to the second member,

$$6x^2 - 3x^2 = 5 + 7,$$

or

$$3x^2 = 12.$$

Dividing by 3,

$$x^2 = 4.$$

The value of x is a number whose square is 4. But

$$2^2 = 4, \text{ and } (-2)^2 = 4.$$

Therefore

$$x = \pm 2.$$

3. This example illustrates the following principle of equivalent equations:

The positive square root of the first member of an equation may be equated in turn to the positive and to the negative square root of the second member.

Ex. 2. Solve the equation $(x-2)(x+2)=-6$.

Simplifying, $x^2-4=-6$.

Transferring -4, $x^2=-2$.

Equating square roots, $x=\pm\sqrt{-2}$.

These results are imaginary. Yet they satisfy the given equation, since

$$(\pm\sqrt{-2}-2)(\pm\sqrt{-2}+2)=(\pm\sqrt{-2})^2-4=-2-4=-6.$$

In such a case the equation is said to have *imaginary roots.* The meaning of an imaginary result, when it arises in connection with a problem, will be explained in Art. 16.

4. The methods used in Ch. VIII. for solving fractional equations which lead to linear equations apply also to fractional equations which lead to quadratic equations.

Ex. 3. Solve the equation $\dfrac{a+x}{b+x}+\dfrac{x-a}{x-b}=0$.

Clearing of fractions,

$$(a+x)(x-b)+(x-a)(b+x)=0,$$

or, $$x^2+ax-bx-ab+x^2-ax+bx-ab=0.$$

Transferring and uniting terms,

$$2x^2=2ab.$$

Dividing by 2 and equating square roots,

$$x=\pm\sqrt{(ab)}.$$

This equation therefore has *irrational roots.*

EXERCISES I.

Solve each of the following equations:

1. $x^2=729$.
2. $x^2-25=144$.
3. $5x^2-27=2x^2$.
4. $\dfrac{3}{x}=\dfrac{x}{27}$.
5. $\dfrac{8x}{81}=\dfrac{9}{2x}$.
6. $\dfrac{x^2-1}{4}=2$.
7. $\dfrac{5x^2+12}{8}=4$.
8. $\dfrac{1}{x^2+1}=\dfrac{1}{5}$.
9. $\dfrac{18}{x^2-1}=6$.
10. $7x^2-8=9x^2-10$.
11. $5+16x^2=11x^2+15$.

12. $5x^2 + 9 + 7x^2 = 8x^2 + 25.$

13. $5(3x^2 + 1) + 81 = 7(5x^2 - 16) + 18.$

14. $\dfrac{5}{2x^2} - \dfrac{4}{3x^2} = \dfrac{7}{12}.$

15. $\dfrac{2 - x^2}{5} - \dfrac{7x^2 + 9}{6} = -\dfrac{37}{15}.$

16. $7 - \dfrac{15 - x}{x^2} = 6 + \dfrac{x + 10}{x^2}.$

17. $\dfrac{11}{x^2} + 5 = 7\left(1 - \dfrac{1}{x^2}\right).$

18. $(7 + 2x)(7 - 2x) = 13.$

19. $(x + \frac{1}{3})(x - \frac{1}{3}) = 11.$

20. $(x-8)(x+5) = 3(3-x).$

21. $(x+2)(x+3) = 5(x+1).$

22. $(x + 3)^2 = 49.$

23. $(3x + 4)^2 - 49 = 576.$

24. $64x^2 - 80x + 25 = 9.$

25. $(5x+4)^2 + (4x-5)^2 = 82.$

26. $\dfrac{x + 5}{5x + 1} = \dfrac{5x + 1}{x + 5}.$

27. $\dfrac{2x - 3}{3x - 2} = \dfrac{3x - 2}{2x - 3}.$

28. $\dfrac{x + 5}{x + 13} = \dfrac{2x + 7}{3x + 18}.$

29. $\dfrac{3x - 4}{4x + 1} = \dfrac{7x - 24}{8x - 19}.$

30. $\dfrac{x + 3}{8} - \dfrac{10}{x + 1} = \dfrac{1}{2}.$

31. $\dfrac{6x}{7} - \dfrac{14 + x^2}{2x + 7} = 3.$

32. $(2x - 3)(3x - 4) - (x - 13)(x - 4) = 40.$

33. $(5x - 7)(3x + 8) - (x - 10)(9 - x) = 1634.$

34. $\dfrac{x - 1}{x + 1} + \dfrac{x + 1}{x - 1} = \dfrac{3x^2 - 7}{x^2 - 1}.$

35. $\dfrac{5}{x - 5} - \dfrac{3}{2x + 3} = \dfrac{132}{77(x - 6)}.$

36. $\dfrac{64}{x + 7} + \dfrac{11}{x - 8} + \dfrac{6}{x + 2} = \dfrac{81}{x + 12}.$

37. $(5 - x)(3 - x)(1 + x) + (5 + x)(3 + x)(1 - x) = 16.$

38. $ax^2 = b^4.$

39. $(a - bx)^2 = c^2.$

40. $ax^2 + b^2 = bx^2 + a^2.$

41. $(x + a)(x - a) = 3a^2.$

42. $m^2x^2 - 4mx + 4 = 9.$

43. $ax^2 + \dfrac{b}{a} = bx^2 + \dfrac{a}{b}.$

44. $\dfrac{a}{a + x} + \dfrac{b}{b + x} = 1.$

45. $\dfrac{a^2}{a^2 + x^2} = \dfrac{b^2}{x^2 - a^2 + b^2}.$

46. $\dfrac{ax - b}{a - bx} = \dfrac{bx + a}{b + ax}.$

47. $\dfrac{x + 1}{x - 1} = \dfrac{a + bx + cx^2}{a - bx + cx^2}.$

Solution by Factoring.

5. The principle on which the solution of an equation by factoring depends was proved in Ch. VI, Art. 43. The methods given in Ch. VI, Arts. 9–13; Ch. XV, Art. 33; and Ch. XVI, Art. 20, enable us to factor any quadratic expression. The roots of the given quadratic equation are the roots of the equations obtained by equating to 0 each of its factors.

Ex. 1. Solve the equation $3x^2 + 5x - 2 = 0$.

Dividing by 3, $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$.

Adding and subtracting $(\frac{5}{2 \times 3})^2, = \frac{25}{36}$, we have

$$x^2 + \tfrac{5}{3}x + \tfrac{25}{36} - \tfrac{25}{36} - \tfrac{2}{3} = 0.$$

or, $$(x + \tfrac{5}{6})^2 - \tfrac{49}{36} = 0.$$

Factoring, $$(x + \tfrac{5}{6} + \tfrac{7}{6})(x + \tfrac{5}{6} - \tfrac{7}{6}) = 0,$$

or, $$(x + 2)(x - \tfrac{1}{3}) = 0,$$

Equating each factor to 0,

$$x + 2 = 0, \text{ whence } x = -2;$$

$$x - \tfrac{1}{3} = 0, \text{ whence } x = \tfrac{1}{3}.$$

Ex. 2. Solve the equation $2x^2 + 2x - 1 = 0$.

Dividing by 2, $x^2 + x - \frac{1}{2} = 0$.

Adding and subtracting $(\frac{1}{2})^2, = \frac{1}{4}$,

$$x^2 + x + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{2} = 0,$$

or $$(x + \tfrac{1}{2})^2 - (\tfrac{1}{2}\sqrt{3})^2 = 0.$$

Factoring, $(x + \frac{1}{2} + \frac{1}{2}\sqrt{3})(x + \frac{1}{2} - \frac{1}{2}\sqrt{3}) = 0$.

Equating factors to 0,

$$x + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} = 0, \quad x + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} = 0.$$

Whence $x = -\frac{1}{2} - \frac{1}{2}\sqrt{3}$, and $-\frac{1}{2} + \frac{1}{2}\sqrt{3}$.

Such roots are usually written $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$.

Ex. 3. Solve the equation $x^2 - 2x + 19 = 0$.

Adding and subtracting $(-1)^2, = 1$,

$$x^2 - 2x + 1 - 1 + 19 = 0,$$

or, since $\quad -1 + 19 = 18 = -(-18) = -(\sqrt{-18})^2,$

$$= -(3\sqrt{-2})^2,$$

$$(x-1)^2 - (3\sqrt{-2})^2 = 0.$$

Factoring, $(x - 1 + 3\sqrt{-2})(x - 1 - 3\sqrt{-2}) = 0.$

Equating factors to 0,

$$x - 1 + 3\sqrt{-2} = 0,\ x - 1 - 3\sqrt{-2} = 0.$$

Whence, $\quad x = 1 \pm 3\sqrt{-2}.$

EXERCISES II.

Solve each of the following equations:

1. $x^2 - 6x + 5 = 0.$
2. $x^2 - 7x + 10 = 0.$
3. $x^2 - 4x - 21 = 0.$
4. $x^2 = 11x + 12.$
5. $3x^2 + 4x + 1 = 0.$
6. $9x^2 - 12x + 4 = 0.$
7. $6x^2 + 13x - 8 = 0.$
8. $11x^2 - 7x - 18 = 0.$
9. $7x^2 - 20x + 8 = 0.$
10. $7 - 12x^2 = 17x.$
11. $20x^2 - 79x + 77 = 0.$
12. $8x^2 + 13x - 82 = 0.$
13. $x^2 - 2x - 1 = 0.$
14. $x^2 - 6x - 71 = 0.$
15. $x^2 - 2x + 2 = 0.$
16. $x^2 - 4x + 13 = 0.$
17. $(x+8)(x+3) = x - 6.$
18. $(x+7)(x-7) = 2(x+50).$
19. $(2x+1)(x+2) = 3x^2 - 4.$
20. $(x-1)(2x+3) = 4x^2 - 22.$
21. $x^2 - 3 = \frac{1}{6}(x-3).$
22. $x(x+5) = 5(40-x) + 27.$
23. $\dfrac{x}{x+120} = \dfrac{14}{3x-10}.$
24. $\dfrac{x+7}{2x+3} = \dfrac{3x-5}{x+3}.$
25. $\dfrac{x+3}{4} - \dfrac{5}{x-6} = \dfrac{x+11}{6}.$
26. $\dfrac{5}{x} + \dfrac{4x+7}{x+1} = -\dfrac{3}{2}.$
27. $\dfrac{3}{x-1} + \dfrac{5}{x-2} = \dfrac{6}{x-3}.$
28. $\dfrac{x+2}{x+3} - \dfrac{x+4}{x+5} = -\dfrac{14}{x+3}.$

29. $\dfrac{9x+1}{9x-3x^2}=\dfrac{x}{21-7x}-\dfrac{x+3}{21x}.$

30. $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}=0.$

31. $\dfrac{5x-1}{x+3}+\dfrac{7x^2-106}{8x^2-72}=-\dfrac{1}{8}.$

32. $\dfrac{x-2}{x+2}+\dfrac{x+2}{x-2}=\dfrac{2(x+3)}{x-3}.$

33. $\dfrac{x+24}{5x^2-5}=\dfrac{x-7}{x+1}-\dfrac{1}{2x-2}.$

34. $\dfrac{4x+67}{40x^2-36}+\dfrac{x}{30x^2-27}=\dfrac{2}{3}.$

35. $x^2+11ax+28a^2=0.$

36. $x^2-14mx+33m^2=0.$

37. $x^2-2ax+a^2-b^2=0.$

38. $x^2-3ax+2a^2-ab-b^2=0.$

39. $x^2-(2m-1)x+m^2-m-6=0.$

40. $x^2-(3a+2b)x+6ab=0.$

41. $ax^2+(a+2)x+2=0.$

42. $bx^2-2(b+c)x+4c=0.$

43. $(a+1)x^2-ax-1=0.$

44. $(a^2+3a-10)x^2-(2a+3)x+1=0.$

45. $x^2-2(a+b)x+(a+b+c)(a+b-c)=0.$

46. $(m-n)x^2-(m+n)x+2n=0.$

47. $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2.$

48. $\dfrac{x-4a}{2a-b}+\dfrac{2a+b}{x}=0.$

49. $\dfrac{a}{x}+\dfrac{x-a}{ab(b-1)}=\dfrac{2}{b}.$

50. $\dfrac{an}{x+4n}-\dfrac{an}{x-4n}=2.$

Solution by Completing the Square.

6. The following examples illustrate the solution of a quadratic equation by the method called *Completing the Square.*

Ex. 1. Solve the equation $x^2-5x+6=0$.

Transferring 6, $\qquad x^2-5x=-6.$

To complete the square in the first member, we add $(-\frac{5}{2})^2$, $=\frac{25}{4}$, to this member, and therefore also to the second. We then have

$$x^2-5x+\tfrac{25}{4}=\tfrac{25}{4}-6=\tfrac{1}{4}.$$

Equating square roots, $x-\frac{5}{2}=\pm\frac{1}{2}$, by Art. 2.

Whence, $x = \frac{5}{2} \pm \frac{1}{2}$.

Therefore the required roots are 3 and 2.

Ex. 2. Solve the equation

$$7x^2 + 5x + 1 = 0.$$

Transferring 1, $7x^2 + 5x = -1$.

Dividing by 7, $x^2 + \frac{5}{7}x = -\frac{1}{7}$.

Adding $(\frac{5}{2\times 7})^2 = \frac{25}{196}$, $x^2 + \frac{5}{7}x + \frac{25}{196} = \frac{25}{196} - \frac{1}{7} = \frac{-3}{196}$.

Equating square roots, $x + \frac{5}{14} = \pm \frac{1}{14}\sqrt{-3}$.

Whence, $x = -\frac{5}{14} \pm \frac{1}{14}\sqrt{-3}$.

Therefore the required roots are

$$-\tfrac{5}{14} + \tfrac{1}{14}\sqrt{-3} \text{ and } -\tfrac{5}{14} - \tfrac{1}{14}\sqrt{-3}.$$

Ex. 3. Solve the equation

$$(a^2 - b^2)x^2 - 2a^2x + a^2 = 0.$$

Transferring a^2, $(a^2 - b^2)x^2 - 2a^2x = -a^2$.

Dividing by $a^2 - b^2$, $x^2 - \dfrac{2a^2x}{a^2 - b^2} = \dfrac{-a^2}{a^2 - b^2}$.

Adding $\left(-\dfrac{a^2}{a^2 - b^2}\right)^2, = \dfrac{a^4}{(a^2 - b^2)^2}$, to both members,

$$x^2 - \frac{2a^2x}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = -\frac{a^2}{a^2 - b^2} + \frac{a^4}{(a^2 - b^2)^2} = \frac{a^2b^2}{(a^2 - b^2)^2}.$$

Equating square roots, $x - \dfrac{a^2}{a^2 - b^2} = \pm \dfrac{ab}{a^2 - b^2}$.

Whence, $x = \dfrac{a^2 \pm ab}{a^2 - b^2}$.

Therefore the required roots are $\dfrac{a}{a - b}$ and $\dfrac{a}{a + b}$.

The preceding examples illustrate the following method of procedure:

Bring the terms in x and x^2 to the first member, and the terms free from x to the second member, uniting like terms.

If the resulting coefficient of x^2 be not $+1$, divide both members by this coefficient.

Complete the square by adding to both members the square of half the coefficient of x.

Equate the positive square root of the first member to the positive and negative square roots of the second member.

Solve the resulting equations.

EXERCISES III.

Solve each of the following equations:

1. $x^2 - 4x + 3 = 0.$ **2.** $x^2 - 5x = -4.$

3. $x^2 + 2x + 1 = 0.$ **4.** $2x^2 - 7x + 3 = 0.$

5. $3x^2 - 53x + 34 = 0.$ **6.** $14x - 49x^2 - 1 = 0.$

7. $x^2 - 4x + 7 = 0.$ **8.** $110x^2 - 21x + 1 = 0.$

9. $x^2 - 2x + 6 = 0.$ **10.** $x^2 - 1 + x(x-1) = x^2.$

11. $(3x-2)(x-1) = 14.$ **12.** $(2x-1)(x-2) = (x+1)^2.$

13. $x + \frac{1}{x} = 5\frac{1}{5}.$ **14.** $x - \frac{1}{x} = 1\frac{1}{2}.$

15. $x - 1 = \frac{12}{x}.$ **16.** $\frac{21}{x} = x - 4.$

17. $\frac{1}{2x} + \frac{1}{3x} = x - \frac{1}{6}.$ **18.** $x + \frac{1}{x} = 7 + \frac{1}{7}.$

19. $\frac{7}{x-4} = x + 2.$ **20.** $2x + 5 = \frac{11}{4x - 11}.$

21. $\frac{x+3}{x+9} = -\frac{x-4}{x-1}.$ **22.** $\frac{x+1}{x+5} = \frac{3x+1}{7x-1}.$

23. $\frac{10}{1-x} + \frac{27}{1-2x} = 5.$ **24.** $\frac{x+3}{x-5} - \frac{2x-4}{x+5} = 2.$

25. $(2x-3)^2 = 8x.$ **26.** $(2x+1)(x+2) = 3x^2 - 4.$

27. $(5x-3)^2 - 7 = 40x - 47.$ **28.** $(x+1)(2x+3) = 4x^2 - 22$

29. $(x-7)(x-4) + (2x-3)(x-5) = 103.$

30. $10(2x+3)(x-3) + (7x+3)^2 = 20(x+3)(x-1).$

31. $(x-1)(x-3) + (x-3)(x-5) = 32.$

32. $(x-1)(x-2) + (x-3)(x-4) = (x-1)^2 - 2.$

33. $\dfrac{6}{x-5}-\dfrac{3}{x-4}=\dfrac{8}{x-3}.$

34. $\dfrac{12}{x+1}-\dfrac{7}{6-x}=-\dfrac{15}{x-2}.$

35. $\dfrac{5x}{x+2}+\dfrac{6}{x+3}+\dfrac{7}{x+4}=5$

36. $\dfrac{2x-7}{2x-1}-\dfrac{7}{5x-4}+\dfrac{11}{3x-4}=1.$

37. $\dfrac{3x}{x^2+3x+2}+\dfrac{6}{x^2+5x+6}=\dfrac{8}{x^2+4x+3}.$

38. $\dfrac{\frac{1}{6}x}{x^2-9x+20}-\dfrac{1}{x^2-7x+10}=\dfrac{2}{x^2-6x+8}.$

39. $\dfrac{x+2}{6x^2+5x+1}+\dfrac{1+x}{10x^2+7x+1}=\dfrac{1-3x}{15x^2+8x+1}.$

40 $x-\dfrac{a}{b}=\dfrac{b}{a}-\dfrac{1}{x}.$

41. $\dfrac{n+x}{n-x}+\dfrac{n-x}{n+x}=\dfrac{n^2}{n^2-x^2}.$

42. $x=\dfrac{3}{(a-b)^2x}-\dfrac{2}{a-b}.$

43. $\dfrac{x^2+1}{n^2x-2n}-\dfrac{1}{2-nx}=\dfrac{x}{n}.$

44. $\dfrac{a-2b}{8x^2-2b^2}=\dfrac{1}{2x+b}-\dfrac{1}{2a}.$

45. $\dfrac{a}{nx-x}-\dfrac{a-1}{x^2-2nx^2+n^2x^2}=1.$

46. $\dfrac{x-a+b}{x+a-b}=\dfrac{a-b-x}{a+b+x}.$

47. $\dfrac{ax}{ax+1}=\dfrac{1-a}{a^2x^2-a-a^2x+ax}.$

48. $\left(\dfrac{a+x}{a-x}\right)^2+\dfrac{7}{2}\cdot\dfrac{a+x}{a-x}+3=0.$

General Solution

7. The most general form of the quadratic equation in one unknown number is evidently

$$ax^2+bx+c=0.$$

The coefficient a is assumed to be *positive* and not 0, but b and c may either or both be positive or negative, or 0.

Dividing by a, $x^2+\frac{b}{a}x+\frac{c}{a}=0.$

Transferring $\frac{c}{a}$, $x^2+\frac{b}{a}x=-\frac{c}{a}.$

Adding $\left(\frac{b}{2a}\right)^2, =\frac{b^2}{4a^2}$, $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$

$$=\frac{b^2-4ac}{4a^2}.$$

Equating square roots, $x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}.$

Whence, $x=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a},$

and $x=-\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$

8. The roots of any quadratic equation can be obtained by substituting in the general solution the particular values of the coefficients a, b, and c.

Ex. Solve the equation $3x^2+7x-10=0$.

We have $a=3,\ b=7,\ c=-10.$

Substituting these values in the general solution, we obtain

$$x=-\tfrac{7}{6}+\tfrac{1}{6}\sqrt{[49-4\times 3(-10)]}=1,$$

and $$x=-\tfrac{7}{6}-\tfrac{1}{6}\sqrt{[49-4\times 3(-10)]}=-\tfrac{10}{3}.$$

EXERCISES IV.

Solve each of the following equations:

1. $2x^2=3x+2.$
2. $5x^2-6x+1=0.$
3. $9x(x+1)=28.$
4. $x^2-b^2=2ax-a^2.$
5. $x^2+6ax+1=0.$
6. $x^2+1=2\tfrac{1}{6}x.$
7. $(x-5)^2+(x-10)^2=37.$
8. $2x(3n-4x)=n^2.$
9. $n^2(x^2+1)=a^2+2n^2x.$
10. $x^2+(x+a)^2=a^2.$

Relation between Roots and Coefficients.

9. If the roots of the quadratic equation

$$ax^2 + bx + c = 0, \text{ or } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

be designated by r_1 and r_2, we have

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}.$$

The sum of the roots is

$$r_1 + r_2 = -\frac{b}{a}. \tag{1}$$

The product of the roots is

$$r_1 r_2 = \left[-\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}\right] \times \left[-\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}\right]$$

$$= \left[-\frac{b}{2a}\right]^2 - \left[\frac{\sqrt{(b^2 - 4ac)}}{2a}\right]^2 = \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{c}{a}. \tag{2}$$

The relations (1) and (2) may be expressed thus:

(i.) *If the coefficient of the second power of the unknown number be* 1, *the sum of the roots is equal to the coefficient of the first power of the unknown number, with sign reversed.*

(ii.) *If the coefficient of the second power of the unknown number be* 1, *the product of the roots is equal to the term free from the unknown number.*

E.g., the roots of the equation $x^2 - 5x + 6 = 0$ are 2 and 3; their sum is 5 (the coefficient of x with sign reversed), and their product is 6 (the term free from x).

The roots of the equation $6x^2 - x - 2 = 0$, or $x^2 - \frac{1}{6}x - \frac{1}{3} = 0$, are $\frac{2}{3}$ and $-\frac{1}{2}$; their sum is $\frac{1}{6}$, and the product is $-\frac{1}{3}$.

10. Formation of an Equation from its Roots. — The relations of the last article enable us to form an equation if its roots be given. We may always assume that the coefficient of the second power of the unknown number is 1.

Ex. 1. Form the equation whose roots are -1, 2.

We have $r_1 + r_2 = -1 + 2 = 1$, the coefficient of x, with sign reversed; and $r_1 r_2 = -1 \times 2 = -2$, the term free from x.

Therefore the required equation is $x^2 - x - 2 = 0$.

Ex. 2. Form the equation whose roots are $1 + 2\sqrt{3}$, $1 - 2\sqrt{3}$.

We have $r_1 + r_2 = (1 + 2\sqrt{3}) + (1 - 2\sqrt{3}) = 2$;

and $$r_1 r_2 = (1 + 2\sqrt{3})(1 - 2\sqrt{3}) = 1 - 12 = -11.$$

Therefore the required equation is $x^2 - 2x - 11 = 0$.

11. It follows from Art. 9, that the quadratic equation may be written in the form

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0,$$

or $$(x - r_1)(x - r_2) = 0.$$

Ex. Form the equation whose roots are -1, 2.

We have $(x + 1)(x - 2) = 0$, or $x^2 - x - 2 = 0$.

When the roots are irrational or imaginary, the method of the preceding article is to be preferred.

EXERCISES V.

Form the equations whose roots are:

1. $8, 2.$ **2.** $-5, -3.$ **3.** $10, 10.$ **4.** $7, -3.$

5. $4, -10.$ **6.** $2\frac{1}{2}, 1\frac{3}{5}.$ **7.** $-\frac{2}{3}, -1\frac{1}{2}.$ **8.** $-\frac{1}{4}, 8.$

9. $2, 0.$ **10.** $a, b.$ **11.** $-a, -1.$ **12.** $a^2, -4a^2.$

13. $\sqrt{2}, -\sqrt{2}.$ **14.** $\frac{1}{2}\sqrt{-3}, -\frac{1}{2}\sqrt{-3}.$

15. $1 + \sqrt{7}, 1 - \sqrt{7}.$ **16.** $\frac{1}{2} - \frac{1}{3}\sqrt{11}, \frac{1}{2} + \frac{1}{3}\sqrt{11}.$

17. $3 - \sqrt{-5}, 3 + \sqrt{-5}.$ **18.** $\frac{2}{3} - \frac{1}{2}\sqrt{-1}, \frac{2}{3} + \frac{1}{2}\sqrt{-1}.$

Nature of the Roots.

12. In many applications it is important to know, without having to solve an equation, the nature of its roots, *i.e.*, whether they are both *real and unequal*, whether they are both *real and equal*, whether they are *imaginary*.

In the general solution

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}, \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

of the equation $\qquad ax^2 + bx + c = 0,$

a, b, and c are limited to real, rational values.

(i.) *The two roots are real and unequal when* $b^2 - 4ac$ *is positive*, i.e., *when* $b^2 - 4ac > 0$.

E.g., in $\qquad x^2 + 4x - 12 = 0,$

$$a = 1,\ b = 4,\ c = -12;\ \text{and since } b^2 - 4ac,\ = 16 + 48,$$

is *positive*, the roots of this equation are real and unequal.

(ii.) *The two roots are real and equal when* $b^2 - 4ac$ *is equal to* 0; i.e., *when* $b^2 = 4ac$.

E.g., in $\qquad x^2 - 4x + 4 = 0,$

$$a = 1,\ b = -4,\ c = 4;\ \text{and since } b^2 = 4ac,$$

the roots of this equation are real and equal.

(iii.) *The two roots are conjugate complex numbers when* $b^2 - 4ac$ *is negative;* i.e., *when* $b^2 - 4ac < 0$.

E.g., in $\qquad x^2 - 2x + 3 = 0,$

$$a = 1,\ b = -2,\ c = 3;\ \text{and since } b^2 - 4ac,\ = 4 - 12,\ = -8,$$

is *negative*, the roots of this equation are complex numbers.

EXERCISES VI.

Without solving the following equations, determine the nature of the roots of each one:

1. $x^2 + 17x + 70 = 0$. 2. $x^2 + 12x = -40$. 3. $x^2 + 5x - 14 = 0$.

4. $x^2 - x = 12$. 5. $x^2 - 8x + 25 = 0$.

6. $x^2 - 8x = 16$. 7. $9x^2 - 12x + 4 = 0$.

8. $8x^2 - 2x - 25 = 0$. 9. $16x^2 + 8x + 49 = 0$.

10. $10x^2 - 21x - 10 = 0$.

For what values of m are the roots of each of the following equations equal? For what values of m are the roots real and unequal? And for what values of m are the roots complex numbers?

11. $mx^2 + 4x + 1 = 0$. **12.** $2x^2 + mx + 1 = 0$.

13. $3x^2 + 6x + m = 0$. **14.** $mx^2 + mx + 1 = 0$.

IRRATIONAL EQUATIONS.

13. An irrational equation may lead to a quadratic equation when rationalized.

Ex. 1. Solve the equation $x + \sqrt{(25 - x^2)} = 7$.

Transferring x, $\sqrt{(25 - x^2)} = 7 - x$. (1)

Squaring, $25 - x^2 = 49 - 14x + x^2$. (2)

The roots of this equation are 3, 4.

Both roots of (2) satisfy the given equation, since

$$3 + \sqrt{(25 - 9)} = 7, \text{ and } 4 + \sqrt{(25 - 16)} = 7.$$

Ex. 2. Solve the equation $x - \sqrt{(25 - x^2)} = 1$.

Transferring x, $-\sqrt{(25 - x^2)} = 1 - x$. (1)

Squaring, $25 - x^2 = 1 - 2x + x^2$. (2)

The roots of this equation are 4 and -3.

The number 4 is a root of the given equation, since

$$4 - \sqrt{(25 - 16)} = 1;$$

but the number -3 is not a root of the given equation, since

$$-3 - \sqrt{(25 - 9)} = -7, \text{ not } 1.$$

Therefore the root -3 was introduced by squaring. Now observe that the same rational equation (2) would have been obtained, if the given equation had been

$$x + \sqrt{(25 - x^2)} = 1; \quad (3)$$

that is, if the surd term had been of opposite sign. The root -3 satisfies equation (3), since

$$-3 + \sqrt{(25 - 9)} = -3 + 4 = 1.$$

Therefore equation (2) is equivalent to equations (1) and (3) jointly.

It frequently happens that no root can be found to satisfy an equation obtained by giving to the square root either its positive or its negative value.

In Ex. 1, the equation thus derived is

$$x-\sqrt{(25-x^2)}=7,$$

and is not satisfied by either of the roots obtained. The equation is then said to be *impossible*.

Ex. 3. Solve the equation

$$\sqrt{(2x+3)}-\sqrt{(7-x)}=1.$$

If both *positive* and *negative* square roots be admitted, the given equation is equivalent to the four equations:

$$\sqrt{(2x+3)}+\sqrt{(7-x)}=1 \ (1), \qquad \sqrt{(2x+3)}-\sqrt{(7-x)}=1 \ (2),$$

$$-\sqrt{(2x+3)}+\sqrt{(7-x)}=1 \ (3), \quad -\sqrt{(2x+3)}-\sqrt{(7-x)}=1 \ (4).$$

The same rational integral equation will evidently be derived by rationalizing any one of these equations.

In (1) transferring $\sqrt{(7-x)}$,

$$\sqrt{(2x+3)}=1-\sqrt{(7-x)}.$$

Squaring, $\quad 2x+3=1-2\sqrt{(7-x)}+7-x,$

or $\quad 3x-5=-2\sqrt{(7-x)}.$

Again squaring, $9x^2-30x+25=28-4x,$

or $\quad 9x^2-26x-3=0.$

The roots of this equation are 3 and $-\frac{1}{9}$. By substitution we find that equation (2) is satisfied by the root 3, and equation (3) by the root $-\frac{1}{9}$. The other two equations are impossible.

Consequently, in solving an irrational equation, we must expect to obtain not only its roots, but also the roots of the other equations obtained by changing the signs of the radicals in all possible ways. Some of these equations will be impossible. The roots of the other irrational equations will be the roots of the rational equation.

14. Ex. Solve the equation

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16.$$

Since $\qquad -3x^2+2x=-(3x^2-2x+4)+4,$

we may take $\sqrt{(3x^2-2x+4)}$ as the unknown number, replacing it temporarily by y. We then have the quadratic equation

$$y-y^2+4=-16.$$

The roots of this equation are 5, and -4.

Equating $\sqrt{(3x^2-2x+4)}$ to each of these roots, we have

$$\sqrt{(3x^2-2x+4)}=5, \text{ whence } x=3, -\tfrac{7}{3}.$$

$$\sqrt{(3x^2-2x+4)}=-4, \text{ whence } x=\tfrac{1}{3}(1\pm\sqrt{37}).$$

The numbers 3, $-\frac{7}{3}$ satisfy the given equation, and are therefore roots of that equation. The numbers $\frac{1}{3}\sqrt{(1\pm\sqrt{37})}$ do not satisfy the given equation.

But if the value of the radical is not restricted to the positive root, the given equation comprises the two equations

$$\sqrt{(3x^2-2x+4)}-3x^2+2x=-16, \tag{1}$$

$$-\sqrt{(3x^2-2x+4)}-3x^2+2x=-16 \tag{2}$$

Then $\frac{1}{3}(1\pm\sqrt{37})$ are roots of (2).

The given equation is said to be in *quadratic form*.

EXERCISES VII.

Solve each of the following equations, and check the results. If a result does not satisfy an equation as written, determine what signs the radical terms must have in order that the result may satisfy the equation.

1. $\sqrt{(x^2-9)}=4.$
2. $4x=3\sqrt{(2x^2-4)}.$
3. $3-\sqrt{(3x^2-4x+9)}=0.$
4. $5x=2\sqrt{(3x^2-x+15)}.$
5. $\sqrt{[(x-5)-7+\sqrt{(x-12)}]}=0.$
6. $\sqrt{[4x-\sqrt{(2x+3)}]}=3.$
7. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}.$
8. $\dfrac{x+\sqrt{(x^2+7)}}{28}=\dfrac{1}{\sqrt{(x^2+7)}}.$
9. $\dfrac{2x+\sqrt{(4x^2-1)}}{2x-\sqrt{(4x^2-1)}}=4.$
10. $\dfrac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}}=\dfrac{5}{11}.$

11. $7\sqrt{x}=3\sqrt{(x^2+3x-59)}$. 12. $\sqrt{(x+2)}-\sqrt{(x^2+2x)}=0$.

13. $(5-\sqrt{x})^2=2(7+\sqrt{x})$. 14. $x+5-\sqrt{(x+5)}=6$.

15. $\sqrt{(x-2)}+2\sqrt{(x+3)}-2\sqrt{(3x-2)}=0$.

16. $\sqrt{(2x+9)}+\sqrt{(3x-15)}=\sqrt{(7x+8)}$.

17. $\sqrt{\dfrac{3x-4}{x-5}}+\sqrt{\dfrac{x-5}{3x-4}}=\dfrac{5}{2}$. 18. $\sqrt{\dfrac{3x+6}{7x-3}}+\sqrt{\dfrac{7x-3}{3x+6}}=\dfrac{13}{6}$.

19. $\dfrac{1}{\sqrt{(x+2)}}+\dfrac{1}{\sqrt{(3x-2)}}=\dfrac{4}{\sqrt{(3x^2+4x-4)}}$.

20. $\dfrac{1}{x-\sqrt{(2-x^2)}}+\dfrac{1}{x+\sqrt{(2-x^2)}}=1$.

21. $x^2-x+2\sqrt{(x^2-x-11)}=14$.

22. $x^2+24=2x+6\sqrt{(2x^2-4x+16)}$.

23. $\sqrt{(2x^2-3x+5)}+2x^2-3x=1$.

24. $\sqrt{(2x^2-7x+7)}+\sqrt{(2x^2+9x-1)}=6$.

25. $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}=\dfrac{a}{b}$. 26. $\dfrac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\dfrac{a}{b}$.

27. $\sqrt{(a+x)}+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a+x)}}$.

28. $\dfrac{x^2}{a-\sqrt{(a^2-x^2)}}-\dfrac{x^2}{a+\sqrt{(a^2-x^2)}}=a$.

29. $\sqrt{(1-x+x^2)}+\sqrt{(1+x+x^2)}=m$.

HIGHER EQUATIONS.

15. Certain equations of higher degree than the second can be solved by means of quadratic equations.

Ex. 1. Solve the equation $\quad x^3-1=0$.

Factoring, $\quad (x-1)(x^2+x+1)=0$.

This equation is equivalent to the two equations

$$x-1=0, \text{ whence } x=1;$$

and $\quad x^2+x+1=0, \text{ whence } x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$

This example gives the three cube roots of 1, since $x^3-1=0$ is equivalent to

$$x^3=1, \text{ or } x=\sqrt[3]{1}.$$

Therefore the three cube roots of 1 are

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3},\ -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

In general, the three cube roots of any number can be found by multiplying the arithmetical cube root of the number in turn by the three algebraic cube roots of 1.

$$\sqrt[3]{8}=2\sqrt[3]{1}=2,\ -1\pm\sqrt{-3}.$$

Ex. 2. Solve the equation $x^4-9=2x^2-1$.

Since $x^4=(x^2)^2$, we may take x^2 as the unknown number and solve this equation as a quadratic in x^2.

We then have $\qquad (x^2)^2-2x^2-8,=0.$

Factoring, $\qquad (x^2-4)(x^2+2)=0.$

Whence,

$x^2-4=0$, or $x=\pm 2$; and $x^2+2=0$, or $x=\pm\sqrt{-2}$.

In general, any equation containing only two powers of the unknown number, *one of which is the square of the other*, can be solved as a quadratic equation.

Ex. 3. Solve the equation $(x^2-3x+1)^2=6+5(x^2-3x+1)$.

In this example x^2-3x+1 is regarded as the unknown number, and may temporarily be represented by the letter y. The equation then becomes

$$y^2=6+5y;\ \text{whence } y=6, \text{ and } -1.$$

We therefore have the two equations

$$x^2-3x+1=6,\ \text{whence } x=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{29};$$

$$x^2-3x+1=-1,\ \text{whence } x=2,\ x=1.$$

Therefore the roots of the given equation are $\frac{3}{2}\pm\frac{1}{2}\sqrt{29}, 2, 1$.

Attention is called to the fact that, in each example, we have obtained as many roots as there are units in the degree of the equation.

EXERCISES VII

Solve each of the following equations:

1. $x^3+1=0$.
2. $x^4-1=0$.
3. $x^6+1=0$.
4. $x^6-1=0$.
5. $(x-1)^3=8$.
6. $x^3=(2a-x)^3$.
7. $(x+1)^4=16$.
8. $x^4+9=10x^2$.
9. $x^4-6x^2=-1$.
10. $x^6-65x^3=-64$.
11. $x^8+5x^4=6$.
12. $(x^2-x+1)^2=3x(x-1)+1$.
13. $(3x^2-5x+1)^2-9x^2+15x=7$.
14. $15x^2-35x-3(7x-3x^2+8)^2+310=0$.
15. $\dfrac{(a+x)^4+(a-x)^4}{(a+x)^3+(a-x)^3}=2a$.
16. $\dfrac{x^4+6x^2+1}{x^4-6x^2+1}=\dfrac{3}{2}$.
17. $\dfrac{x^2-a^2}{x^2+a^2}+\dfrac{x^2+a^2}{x^2-a^2}=\dfrac{34}{15}$.
18. $\dfrac{x^2-5x+3}{x^2+5x-3}-\dfrac{x^2+5x-3}{x^2-5x+3}=\dfrac{8}{3}$.

PROBLEMS.

16. Pr. **1.** The sum of two numbers is 15, and their product is 56. What are the numbers?

Let x stand for one of the numbers; then, by the first condition, $15-x$ stands for the other number. By the second condition

$$x(15-x)=56;\ \text{whence } x=7, \text{ and } 8.$$

Therefore $x=7$, one of the numbers, and $15-x=8$, the other number. Observe that if we take $x=8$, then $15-x=7$. That is, the two required numbers are the two roots of the quadratic equation.

Pr. **2.** Divide 100 into two parts whose product is 2600.

Let x stand for the less part, and $100-x$ for the greater.

By the second condition, $x(100-x)=2600$. The roots of this equation are $50+10\sqrt{-1}$ and $50-10\sqrt{-1}$.

An imaginary result always indicates inconsistent conditions in the problem. The inconsistency of these conditions may be shown as follows:

Let d stand for the difference between the two parts of 100. Then $50 + \frac{1}{2}d$ stands for the greater part, and $50 - \frac{1}{2}d$ for the less.

The product of the two parts is

$$(50 + \tfrac{1}{2}d)(50 - \tfrac{1}{2}d), = 2500 - (\tfrac{1}{2}d)^2 = 2500 - \tfrac{1}{4}d^2.$$

Since d^2 is positive for all *real* values of d, the product $2500 - \frac{1}{4}d^2$ must be less than 2500. Consequently 100 cannot be divided into two parts whose product is greater than 2500.

17. When the solution of a problem leads to a quadratic equation, it is necessary to determine whether either or both of the roots of the equation satisfy the conditions expressed and implied in the problem.

Positive results, in general, satisfy all the conditions of the problem.

A *negative result,* as a rule, satisfies the conditions of the problem, when they refer to abstract numbers. When the required numbers refer to quantities which can be understood in opposite senses, as opposite directions, etc., an intelligible meaning can usually be given to a negative result.

An *imaginary result* always implies inconsistent conditions.

18. The interpretation of a negative result is often facilitated by the following principle:

If a given quadratic equation have a negative root, then the equation obtained by changing the sign of x has a positive root of the same absolute value.

E.g., the roots of the equation $x^2 - 5x + 6 = 0$ are 2 and 3: and the roots of the equation

$$(-x)^2 - 5(-x) + 6 = 0,$$

or $$x^2 + 5x + 6 = 0, \text{ are } -2 \text{ and } -3.$$

Pr. 3. A man bought muslin for \$3.00. If he had bought 3 yards more for the same money, each yard would have cost him 5 cents less. How many yards did he buy?

Let x stand for the number of yards the man bought. Then 1 yard cost $\frac{300}{x}$ cents. If he had bought $x+3$ yards for the same money, each yard would have cost $\frac{300}{x+3}$ cents.

Therefore $\frac{300}{x}-\frac{300}{x+3}=5$; whence $x=12$ and -15.

The root 12 satisfies the equation and also the conditions of the problem; the root -15 has no meaning.

But if x be replaced by $-x$ in the equation, we obtain a new equation,

$$\frac{300}{-x}-\frac{300}{-x+3}=5, \text{ or } \frac{300}{x-3}-\frac{300}{x}=5, \tag{1}$$

whose roots are -12 and $+15$.

Equation (1) evidently corresponds to the problem: A man bought muslin for \$3.00. If he had bought 3 yards less for the same money, each yard would have cost him 5 cents more.

Notice that the intelligible result, 12, of the first statement has become -12 and is meaningless in the second statement.

EXERCISES IX.

1. If 1 be added to the square of a number, the sum will be 50. What is the number?

2. If 5 be subtracted from a number, and 1 be added to the square of the remainder, the sum will be 10. What is the number?

3. One of two numbers exceeds 50 by as much as the other is less than 50, and their product is 2400. What are the numbers?

4. The product of two consecutive integers exceeds the smaller by 17,424. What are the numbers?

5. If 27 be divided by a certain number, and the same number be divided by 3, the results will be equal. What is the number?

6. What number, added to its reciprocal, gives 2.9?

7. What number, subtracted from its reciprocal, gives n? Let $n = 6.09$.

8. If n be divided by a certain number, the result will be the same as if the number were subtracted from n. What is the number? Let $n = 4$.

9. If the product of two numbers be 176, and their difference be 5, what are the numbers?

10. A certain number was to be added to $\frac{1}{2}$, but by mistake $\frac{1}{2}$ was divided by the number. Nevertheless, the correct result was obtained. What was the number?

11. If 100 marbles be so divided among a certain number of boys that each boy shall receive four times as many marbles as there are boys, how many boys are there?

12. The area of a rectangle, one of whose sides is 7 inches longer than the other, is 494 square inches. How long is each side?

13. The difference between the squares of two consecutive numbers is equal to three times the square of the less number. What are the numbers?

14. A merchant received \$ 48 for a number of yards of cloth. If the number of dollars a yard be equal to three-sixteenths of the number of yards, how many yards did he sell?

15. In a company of 14 persons, men and women, the men spent \$ 24 and the women \$ 24. If each man spent \$ 1 more than each woman, how many men and how many women were in the company?

16. A pupil was to add a certain number to 4, then to subtract the same number from 9, and finally to multiply the results. But he added the number to 9, then subtracted 4 from the number, and multiplied these results. Nevertheless he obtained the correct product. What was the number?

17. A man paid \$ 80 for wine. If he had received 4 gallons less for the same money, he would have paid \$ 1 more a gallon. How many gallons did he buy?

18. A man left \$ 31,500 to be divided equally among his children. But since 3 of the children died, each remaining child received \$ 3375 more. How many children survived?

19. Two bodies move from the vertex of a right angle along its sides at the rate of 12 feet and 16 feet a second respectively. After how many seconds will they be 90 feet apart?

20. A tank can be filled by two pipes, by the one in two hours less time than by the other. If both pipes be open $1\frac{7}{8}$ hours, the tank will be filled. How long does it take each pipe to fill the tank?

21. From a thread, whose length is equal to the perimeter of a square, 36 inches are cut off, and the remainder is equal in length to the perimeter of another square whose area is four-ninths of that of the first. What is the length of the thread?

22. A number of coins can be arranged in a square, each side containing 51 coins. If the same number of coins be arranged in two squares, the side of one square will contain 21 more coins than the side of the other. How many coins does the side of each of the latter squares contain?

23. A farmer wished to receive \$ 2.88 for a certain number of eggs. But he broke 6 eggs, and in order to receive the desired amount he increased the price of the remaining eggs by $2\frac{2}{5}$ cents a dozen. How many eggs had he originally?

24. Two bodies move toward each other from A and B respectively, and meet after 35 seconds. If it takes the one 24 seconds longer than the other to move from A to B, how long does it take each one to move that distance?

25. It takes a boat's crew 4 hours and 12 minutes to row 12 miles down a river with the current, and back again against the current. If the speed of the current be 3 miles an hour, at what rate can the crew row in still water?

26. A man paid \$ 300 for a drove of sheep. By selling all but 10 of them at a profit of \$ 2.50 each, he received the amount he paid for all the sheep. How many sheep did he buy?

J. Frenck

CHAPTER XIX.

SIMULTANEOUS QUADRATIC AND HIGHER EQUATIONS.

1. The solution of a system of quadratic or higher equations in general involves the solution of an equation of higher degree than the second, and therefore cannot be effected by the methods for solving quadratic equations. But there are many special systems whose solutions can be made to depend upon the solutions of quadratic equations.

The following methods are based upon equivalent systems of equations.

2. Elimination by Substitution. — When one equation of a system of two equations is of the first degree, the solution can be obtained by the method of substitution.

Ex. Solve the system
$$\left.\begin{aligned} y + 2x &= 5, \\ x^2 - y^2 &= -8. \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Solving (1) for y, $\quad y = 5 - 2x. \quad (3)$

Substituting $5 - 2x$ for y in (2),
$$x^2 - 25 + 20x - 4x^2 = -8. \quad (4)$$

From this equation we obtain $\quad x = 1,$

and $\quad x = 5\frac{2}{3}.$

Substituting 1 for x in (3), $\quad y = 3.$

Substituting $5\frac{2}{3}$ for x in (3), $\quad y = -6\frac{1}{3}.$

The equations (3)–(4) are equivalent to the given equations (1)–(2).

Therefore the solutions of the given system are 1, 3; $5\frac{2}{3}$, $-6\frac{1}{3}$, the first number of each pair being the value of x, and the second the corresponding value of y.

Had we substituted 1 for x in (2), we should have obtained $y = \pm 3$.

But the solution 1, -3 does not satisfy equation (1).

Therefore, always substitute in the linear equation the value of the unknown number obtained by elimination.

3. Elimination by Addition and Subtraction.—This method can frequently be applied.

Ex. Solve the system

$$\left.\begin{aligned} x^2 + 3y &= 18, \\ 2x^2 - 5y &= 3. \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

We will first eliminate y.

Multiplying (1) by 5, $\qquad 5x^2 + 15y = 90. \qquad (3)$

Multiplying (2) by 3, $\qquad 6x^2 - 15y = 9. \qquad (4)$

Adding (3) and (4), $\qquad 11x^2 = 99.$

Whence, $\qquad x = 3$, and $x = -3$.

Substituting 3 for x in (1), $\qquad y = 3$.

Substituting -3 for x in (1), $\qquad y = 3$.

The given system has the two solutions 3, 3; -3, 3.

Notice that this example could also have been solved by the method of substitution.

EXERCISES I.

Solve each of the following systems:

1. $\begin{cases} xy = 54, \\ 3x = 2y. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 13, \\ x^2 - y^2 = 5. \end{cases}$

3. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

4. $\begin{cases} 4x - 3y = 24, \\ xy = 96. \end{cases}$

5. $\begin{cases} 2x^2 - 3y^2 = 24, \\ 2x = 3y. \end{cases}$

6. $\begin{cases} 2x^2 - 3y = 20, \\ x^2 + 5y = 36. \end{cases}$

7. $\begin{cases} 3x - 2y = 1, \\ x^2 + y^2 = 74. \end{cases}$

8. $\begin{cases} 7x + xy = 20, \\ 2xy + 5x = 22. \end{cases}$

9. $\begin{cases} 2x + 3y = 10, \\ x(x + y) = 25. \end{cases}$

10. $\begin{cases} 4x^2 - xy = 0, \\ 2x - 3y = 6. \end{cases}$

11. $\begin{cases} 5xy + 3x^2 = 132, \\ 5xy - 3x^2 = 78. \end{cases}$

12. $\begin{cases} 4x = xy + 5, \\ 7y = xy + 6. \end{cases}$

13. $\begin{cases} 3x = x^2 + y^2 - 1, \\ 3y = x^2 + y^2 - 7. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 343, \\ 2x - y = 21. \end{cases}$

15. $\begin{cases} 2x^2 - 3xy + y^2 = 14, \\ 2x - y = 7. \end{cases}$

16. $\begin{cases} x^2 + 5xy + y^2 = 43, \\ x^2 + 5xy - y^2 = 25. \end{cases}$

17. $\begin{cases} 2x - 3y = 11, \\ \dfrac{4}{x} - \dfrac{3}{y} = -\dfrac{17}{7}. \end{cases}$

18. $\begin{cases} x + 2y = 1, \\ \dfrac{x}{y} + \dfrac{y}{x} + 3\frac{1}{8} = 0. \end{cases}$

19. $\begin{cases} \dfrac{x+y}{x-y} + 3x = 2\frac{2}{3}, \\ 5\dfrac{x+y}{x-y} - 7x = -8\frac{2}{3}. \end{cases}$

20. $\begin{cases} 3x + \sqrt{\dfrac{x}{y}} = 30, \\ 5x - 2\sqrt{\dfrac{x}{y}} = 39. \end{cases}$

4. Homogeneous Equations. — When all the terms which contain the unknown numbers in both equations of the system are of the second degree, a system can always be derived whose solution is obtained by the method of Art. 2.

Ex. Solve the system
$$x^2 + xy + 2y^2 = 74, \qquad (1)$$
$$2x^2 + 2xy + y^2 = 73. \qquad (2)$$

Multiplying (1) by 73, $73x^2 + 73xy + 146y^2 = 74 \times 73. \qquad (3)$

Multiplying (2) by 74, $148x^2 + 148xy + 74y^2 = 74 \times 73. \qquad (4)$

Subtracting (3) from (4), $75x^2 + 75xy - 72y^2 = 0,$

or $$25x^2 + 25xy - 24y^2 = 0,$$

or $$(5x - 3y)(5x + 8y) = 0.$$

Therefore the given system is equivalent to

$$\left.\begin{aligned} 5x - 3y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\}(a), \qquad \left.\begin{aligned} 5x + 8y &= 0, \\ x^2 + xy + 2y^2 &= 74, \end{aligned}\right\}(b).$$

The solutions of these systems, and hence of the given system, are respectively 3, 5; -3, -5; 8, -5; -8, 5.

In applying this method to such systems, we must first derive from the given equations a homogeneous equation in which there is no term free from the unknown numbers.

5. Such examples can also be solved by a special device.

Ex. Solve the system $x^2 + 4y^2 = 13,$ (1)

$xy + 2y^2 = 5.$ (2)

In both equations, let $y = tx.$ (3)

Then from (1), $x^2 + 4x^2t^2 = 13$, whence $x^2 = \dfrac{13}{1+4t^2}$; (4)

and from (2), $x^2t + 2x^2t^2 = 5$, whence $x^2 = \dfrac{5}{t+2t^2}.$ (5)

Equating values of x^2, $\dfrac{13}{1+4t^2} = \dfrac{5}{t+2t^2}.$ (6)

Whence $t = \frac{1}{3}$, and $t = -\frac{5}{2}$.

When $t = \frac{1}{3}$, $x^2 = \dfrac{13}{1+4t^2} = 9$, whence $x = \pm 3$.

When $t = -\frac{5}{2}$, $x^2 = \frac{1}{2}$, whence $x = \pm\sqrt{\frac{1}{2}}$.

When $x = \pm 3$, $y = tx = \frac{1}{3}(\pm 3) = \pm 1$.

When $x = \pm\sqrt{\frac{1}{2}}$, $y = -\frac{5}{2}(\pm\sqrt{\frac{1}{2}}) = \mp\frac{5}{2}\sqrt{\frac{1}{2}}$.

EXERCISES II.

Solve each of the following systems:

1. $\begin{cases} x^2 + xy = 78, \\ y^2 - xy = 7. \end{cases}$

2. $\begin{cases} x^2 + 4y^2 = 13, \\ xy + 2y^2 = 5. \end{cases}$

3. $\begin{cases} x^2 + xy + y^2 = 52, \\ xy - x^2 = 8. \end{cases}$

4. $\begin{cases} x^2 - xy + y^2 = 21, \\ y^2 - 2xy + 15 = 0. \end{cases}$

5. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$

6. $\begin{cases} x^2 - 2xy + 3y^2 = 9, \\ x^2 - 4xy + 5y^2 = 5. \end{cases}$

7. $\begin{cases} x^2 + xy + y^2 = 13x, \\ x^2 - xy + y^2 = 7x. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 61 - 3xy, \\ x^2 - y^2 = 31 - 2xy. \end{cases}$

6. Symmetrical Equations. — A **Symmetrical Equation** is one which remains the same when the unknown numbers are interchanged.

A system of two symmetrical equations can be solved by first finding the values of $x + y$ and $x - y$.

Ex. 1. Solve the system
$$\left.\begin{aligned} x^2+y^2&=13,\\ xy&=6.\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Multiplying (2) by 2, $\quad 2\,xy=12.\quad$ (3)

Adding (3) to (1), $\quad x^2+2\,xy+y^2=25.\quad$ (4)

Subtracting (3) from (1), $\quad x^2-2\,xy+y^2=1.\quad$ (5)

Equating square roots of (4), $\quad x+y=\pm 5.\quad$ (6)

Equating square roots of (5), $\quad x-y=\pm 1.\quad$ (7)

Equations (4)–(5), or (6)–(7), are equivalent to (1)–(2). But (6) and (7) are equivalent to

$$\left.\begin{aligned}x+y&=5,\\x-y&=1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=5,\\x-y&=-1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-5,\\x-y&=+1,\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-5,\\x-y&=-1.\end{aligned}\right\}$$

The solutions of these four systems are respectively 3, 2; 2, 3; -2, -3; -3, -2.

The solutions of (6) and (7) should be obtained mentally, without writing the equivalent systems. Each sign of the second member of (6) should be taken in turn with each sign of the second member of (7).

Notice that these solutions differ only in having the values or x and y interchanged. This we should expect from the definition of symmetrical equations.

When the equations are symmetrical, except for sign, the solution can be obtained by a similar method.

Ex. 2. Solve the system

$$x-y=3,\qquad (1)$$

$$x^2+y^2=29.\qquad (2)$$

Squaring (1), $\quad x^2-2\,xy+y^2=9,\quad$ (3)

Subtracting (3) from (2),

$$2\,xy=20, \text{ or } xy=10.\qquad (4)$$

The solutions of (1) and (4) are 5, 2; -2, -5.

Notice that the solutions in this case differ not only in having the values of x and y interchanged, but also in sign.

EXERCISES III.

Solve each of the following systems:

1. $\begin{cases} x+y=12, \\ xy=32. \end{cases}$ **2.** $\begin{cases} x+y=a, \\ xy=b. \end{cases}$ **3.** $\begin{cases} \frac{1}{2}x+5y=37, \\ xy=28. \end{cases}$

4. $\begin{cases} x-y=8, \\ xy=-15. \end{cases}$ **5.** $\begin{cases} x-y=m, \\ xy=n. \end{cases}$ **6.** $\begin{cases} 6x-7y=58, \\ 3xy=-60. \end{cases}$

7. $\begin{cases} x^2+y^2=40, \\ xy=12. \end{cases}$ **8.** $\begin{cases} x^2+y^2=181, \\ xy=-90. \end{cases}$ **9.** $\begin{cases} 25x^2+9y^2=148, \\ 5xy=8. \end{cases}$

10. $\begin{cases} 9x^2+y^2=37a^2, \\ xy=-2a^2. \end{cases}$ **11.** $\begin{cases} 5x^2+2y^2=5a^2+8b^2, \\ xy=2ab. \end{cases}$

12. $\begin{cases} x^2+y^2=137, \\ x+y=15. \end{cases}$ **13.** $\begin{cases} x^2+y^2=61, \\ x+y=11. \end{cases}$ **14.** $\begin{cases} 5x+3y=11, \\ 25x^2+9y^2=73. \end{cases}$

15. $\begin{cases} x^2-y^2=28, \\ xy=48. \end{cases}$ **16.** $\begin{cases} x^2-4y^2=-3, \\ xy=-1. \end{cases}$ **17.** $\begin{cases} x^2+y^2=53, \\ x-y=5. \end{cases}$

18. $\begin{cases} x^2+y^2=74, \\ x-y=2. \end{cases}$ **19.** $\begin{cases} 9x^2+y^2=82, \\ 3x-y=10. \end{cases}$ **20.** $\begin{cases} 16x^2+49y^2=113, \\ 4x+7y=1. \end{cases}$

21. $\begin{cases} xy=80, \\ \frac{1}{x}-\frac{1}{y}=\frac{1}{5}. \end{cases}$ **22.** $\begin{cases} x+y=16, \\ \frac{1}{x}+\frac{1}{y}=\frac{1}{3}. \end{cases}$ **23.** $\begin{cases} x^2+y^2=2\frac{1}{2}xy, \\ \frac{1}{x}+\frac{1}{y}=1\frac{1}{2}. \end{cases}$

24. $\begin{cases} \frac{1}{x}+\frac{1}{y}=3, \\ \frac{1}{xy}=2. \end{cases}$ **25.** $\begin{cases} \frac{x}{y}-\frac{y}{x}=\frac{16}{15}, \\ x-y=2. \end{cases}$ **26.** $\begin{cases} \frac{1}{x}+\frac{1}{y}=10, \\ \frac{1}{x^2}+\frac{1}{y^2}=58. \end{cases}$

27. $\begin{cases} x+xy+y=29, \\ x^2+xy+y^2=61. \end{cases}$ **28.** $\begin{cases} x^2+y^2+7xy=171, \\ xy=2(x+y). \end{cases}$

29. $\begin{cases} x^2+y^2-(x-y)=20, \\ xy+x-y=1. \end{cases}$ **30.** $\begin{cases} x^2+y^2-x-y=22, \\ x+y+xy=-1. \end{cases}$

31. $\begin{cases} x^2+y^2+x-y=a, \\ xy+x-y=b. \end{cases}$ **32.** $\begin{cases} x+y=2, \\ x^2+y^2+xy=3. \end{cases}$

33. $\begin{cases} x+y=9, \\ x^2+y^2-xy=21. \end{cases}$ **34.** $\begin{cases} x^2+xy+y^2=2m, \\ x^2-xy+y^2=2n. \end{cases}$

7. Higher Equations. — The solutions of certain equations of higher degree than the second can be made to depend upon the solutions of quadratic equations.

Ex. 1. Solve the system $x^3 + y^3 = 9,$ (1)

$$x + y = 3. \quad (2)$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

Subtracting (3) from the square of (2),

$$3xy = 6, \text{ or } xy = 2. \quad (4)$$

The solutions of (2) and (4), and therefore of the given system, are 1, 2, and 2, 1.

Ex. 2. Solve the system $x^4 + y^4 = 17,$ (1)

$$x + y = 3. \quad (2)$$

We first find the value of xy.

Let $xy = z.$ (3)

Squaring (2), $x^2 + 2xy + y^2 = 9,$ (4)

or $x^2 + y^2 = 9 - 2z.$ (5)

Squaring (5), $x^4 + 2x^2y^2 + y^4 = 81 - 36z + 4z^2,$ (6)

or $x^4 + y^4 = 81 - 36z + 2z^2.$ (7)

Since $x^4 + y^4 = 17$, we have from (7),

$$2z^2 - 36z + 81 = 17. \quad (8)$$

Whence $z = 16$, and 2. (9)

Therefore, from (3) and (9), $xy = 16,$ (10)

and $xy = 2.$ (11)

The solutions of (2) and (10) and of (2) and (11) are readily found, and should be checked by substitution.

EXERCISES IV.

Solve each of the following systems:

1. $\begin{cases} x + y = 5, \\ x^3 + y^3 = 35. \end{cases}$

2. $\begin{cases} x - y = 1, \\ x^3 - y^3 = 7. \end{cases}$

3. $\begin{cases} 2(x + y) = 5, \\ 32(x^3 + y^3) = 2285. \end{cases}$

4. $\begin{cases} (x-1)^3 + (y-2)^3 = 28, \\ x + y = 7. \end{cases}$

5. $\begin{cases}(x-7)^3+(5-y)^3=9,\\ x-y=5.\end{cases}$

6. $\begin{cases}x^4-y^4=544,\\ x^2+y^2=34.\end{cases}$

7. $\begin{cases}x^4+y^4=82,\\ xy=3.\end{cases}$

8. $\begin{cases}x^4+y^4=97,\\ x+y=5.\end{cases}$

9. $\begin{cases}x^4+y^4=257,\\ x-y=3.\end{cases}$

10. $\begin{cases}(x-7)^4+(y-3)^4=257,\\ x-y+1=0.\end{cases}$

11. $\begin{cases}(x^2-y^2)(x+y)=9,\\ xy(x+y)=6.\end{cases}$

12. $\begin{cases}(x+y)(x^2+y^2)=175,\\ (x-y)(x^2-y^2)=7.\end{cases}$

13. $\begin{cases}x^4+y^4=14\,x^2y^2,\\ x+y=a.\end{cases}$

14. $\begin{cases}x^3y^2-x^2y^3=1152,\\ x^2y-xy^2=48.\end{cases}$

Problems.

8. Pr. The front wheel of a carriage makes 6 more revolutions than the hind wheel in travelling 360 feet. But if the circumference of each wheel were 3 feet greater, the front wheel would make only 4 revolutions more than the hind wheel in travelling the same distance as before. What are the circumferences of the two wheels?

Let x stand for the number of feet in the circumference of front wheel, and y for the number of feet in the circumference of hind wheel. Then in travelling 360 feet the front wheel makes $\frac{360}{x}$ revolutions, and the hind wheel makes $\frac{360}{y}$ revolutions.

By the first condition, $\frac{360}{x}=\frac{360}{y}+6.$ (1)

If 3 feet were added to the circumference of each wheel, the front wheel would make $\frac{360}{x+3}$ revolutions, and the hind wheel $\frac{360}{y+3}$ revolutions.

By the second condition, $\frac{360}{x+3}=\frac{360}{y+3}+4.$ (2)

Whence $x=12$, the circumference of the front wheel, and $y=15$, the circumference of the hind wheel.

EXERCISES V.

1. The square of one number increased by ten times a second number is 84, and is equal to the square of the second number increased by ten times the first. What are the numbers?

2. The sum of two numbers is 20, and the sum of the square of the one diminished by 13 and the square of the other increased by 13 is 272. What are the numbers?

3. Find two numbers such that their difference added to the difference of their squares shall be 150, and their sum added to the sum of their squares shall be 330.

4. Find two numbers whose sum is equal to their product and also to the difference of their squares.

5. The sum of the fourth powers of two numbers is 1921, and the sum of their squares is 61. What are the numbers?

6. If a number of two digits be multiplied by its tens' digit, the product will be 390. If the digits be interchanged and the resulting number be multiplied by its tens' digit, the product will be 280. What is the number?

7. If a number of two digits be divided by the product of its digits, the quotient will be 2. If 27 be added to the number, the sum will be equal to the number obtained by interchanging the digits. What is the number?

8. The product of the two digits of a number is equal to one-half of the number. If the number be subtracted from the number obtained by interchanging the digits, the remainder will be equal to three-halves of the product of the digits of the number. What is the number?

9. If the difference of the squares of two numbers be divided by the first number, the quotient and the remainder will each be 5. If the difference of the squares be divided by the second number, the quotient will be 13 and the remainder 1. What are the numbers?

10. The sum of the three digits of a number is 9. If the digits be written in reverse order, the resulting number will exceed the original number by 396. The square of the middle digit exceeds the product of the first and third digit by 4. What is the number?

11. A rectangular field is 119 yards long and 19 yards wide. How many yards must be added to its width and how many yards must be taken from its length, in order that its area may remain the same while its perimeter is increased by 24 yards?

12. The floor of a room contains $30\frac{1}{3}$ square yards; one wall contains 21 square yards, and an adjacent wall contains 13 square yards. What are the dimensions of the room?

13. A merchant bought a number of pieces of cloth of two different kinds. He bought of each kind as many pieces and paid for each yard half as many dollars as that kind contained yards. He bought altogether 19 pieces and paid for them \$ 921.50. How many pieces of each kind did he buy?

14. The diagonal of a rectangle is $20\frac{2}{5}$ feet. If the length of one side be increased by 14 feet and the length of the other side be diminished by $2\frac{2}{5}$ feet, the diagonal will be increased by $12\frac{2}{5}$ feet. What are the lengths of the sides of the rectangle?

15. A certain number of coins can be arranged in the form of one square, and also in the form of two squares. In the first arrangement each side of the square contains 29 coins, and in the second arrangement one square contains 41 more coins than the other. How many coins are there in a side of each square of the second arrangement?

16. A piece of cloth after being wet shrinks in length by one-eighth and in breadth by one-sixteenth. The piece contains after shrinking 3.68 fewer square yards than before shrinking, and the length and breadth together shrink 1.7 yards. What was the length and breadth of the piece?

17. A merchant paid \$ 125 for two kinds of goods. He sold the one kind for \$ 91 and the other for \$ 36. He thereby

gained as much per cent on the first kind as he lost on the second. How much did he pay for each kind?

18. Two workmen can do a piece of work in 6 days. How long will it take each of them to do the work, if it takes one 5 days longer than the other?

19. Two men, A and B, receive different wages. A earns \$ 42, and B \$ 40. If A had received B's wages a day, and B had received A's wages, they would have earned together \$ 4 more. How many days does each work, if A works 8 days more than B, and what wages does each receive?

20. It takes a number of workmen 8 hours to remove a pile of stones from one place to another. Had there been 8 more workmen, and had each one carried 5 pounds less at each trip, they would have completed the work in 7 hours. Had there been 8 fewer workmen and had each one carried 11 pounds more at each trip, they would have completed the work in 9 hours. How many workmen were there and how many pounds did each one carry at every trip?

21. A tank can be filled by one pipe and emptied by another. If, when the tank is half full of water, both pipes be left open 12 hours, the tank will be emptied. If the pipes be made smaller, so that it will take the one pipe one hour longer to fill the tank and the other one hour longer to empty it, the tank, when half full of water, will then be emptied in $15\frac{3}{4}$ hours. In what time will the empty tank be filled by the one pipe, and the full tank be emptied by the other?

J. Frencik

CHAPTER XX.

RATIO, PROPORTION, AND VARIATION.

RATIO.

1. The **Ratio** of one number to another is the relation between the numbers which is expressed by the quotient of the first divided by the second.

E.g., the ratio of 6 to 4 is expressed by $\frac{6}{4}$, $=\frac{3}{2}$.

The ratio of one number to another is frequently expressed by placing a colon between them; as $5:7$.

The first number in a ratio is called the **First Term**, or the **Antecedent** of the ratio, and the second number the **Second Term**, or the **Consequent** of the ratio.

Thus, in the ratio $a:b$, a is the first term, and b the second.

2. Since, by definition, a ratio is a fraction, all the properties of fractions are true of ratios; as $a:b=ma:mb$.

3. The definition given in Art. 1 has reference to the ratio of one *number* to another. But it is frequently necessary to compare concrete quantities, as the length of one line with the length of another line, etc.

If two concrete quantities of the same kind can be expressed by two rational numbers in terms of the same unit, then the ratio of the one quantity to the other is defined as the ratio of the one number to the other.

E.g., the ratio of $2\frac{1}{2}$ yards to $1\frac{1}{7}$ yards is $2\frac{1}{2}:1\frac{1}{7}$, $=\dfrac{2\frac{1}{2}}{1\frac{1}{7}}=\dfrac{35}{16}$.

Observe that by this definition the ratio of two concrete quantities is a number. Also that the quantities to be compared must be of the same kind. Dollars cannot be compared with pounds, etc.

4. If two concrete quantities cannot be expressed by two rational numbers, integers or fractions, in terms of the same unit, they are said to be **Incommensurable** one to the other.

Thus, if the lengths of the two sides of a right triangle be equal, the length of the hypothenuse cannot be expressed by a rational number in terms of a side as a unit, or any fraction of a side as a unit.

If a side be taken as the unit, the hypothenuse is expressed by $\sqrt{2}$, an irrational number. And the ratio of the hypothenuse to a side is $\sqrt{2}:1, =\sqrt{2}$. But as was shown in Ch. XV, Art. 40, an approximate value of $\sqrt{2}$ can be found to any required degree of accuracy.

5. In general let P and Q be two incommensurable quantities. Then two rational numbers $\frac{m}{n}$ and $\frac{m+1}{n}$ can be found, between which the value of the ratio $P:Q$ lies. These two fractions differ by $\frac{1}{n}$. Therefore, the ratio $P:Q$, which lies between them, differs from either of them by less than $\frac{1}{n}$. By taking n sufficiently great we can make $\frac{1}{n}$ as small as we please, that is, *less than any assigned number, however small.*

It can be proved that the ratio of two incommensurable quantities is a number which obeys the fundamental laws of algebra.

It is therefore not necessary, in the principles of this chapter, to make any distinction between such ratios and those which can be expressed exactly in terms of integers and fractions.

EXERCISES I.

What is the ratio of

1. $6a$ to $9b$? **2.** $\frac{3}{5}a^2b$ to $\frac{6}{11}ab^2$? **3.** $9\frac{1}{2}x^3y$ to $7\frac{3}{5}xy^5$?

4. $\frac{1}{a}$ to $\frac{1}{b}$? **5.** $\frac{a}{b}$ to $\frac{c}{d}$? **6.** $\frac{a}{x-3}$ to $\frac{1}{(x-3)^2}$?

7. Which is the greater ratio,

$$a+2b:a+b \text{ or } a+3b:a+2b?$$

What is the value of the ratio $x:y$

8. If $\dfrac{6x+2y}{3x-y}=10$? 9. If $\dfrac{8x+4y}{3x-2y}=5$?

If the value of the ratio $x:y$ is $\frac{3}{5}$, what is the value

10. Of $\dfrac{10x-y}{15x+y}$? 11. Of $\dfrac{5x+6y}{3x-2y}$?

PROPORTION.

6. A **Proportion** is an equation whose members are two equal ratios.

E.g., $4:3=8:6$, read *the ratio of* 4 *to* 3 *is equal to the ratio of* 8 *to* 6, or 4 *is to* 3 *as* 8 *is to* 6.

Instead of the equality sign a double colon is frequently used; as $4:3::8:6$.

7. Four numbers are said to be *in proportion*, or to be *proportional*, when the first is to the second as the third is to the fourth.

E.g., the numbers 4, 3, 8, 6 are proportional, since $4:3=8:6$.

The individual numbers are called the **Proportionals**, or **Terms** of the proportion.

The **Extremes** of a proportion are its first and last terms; as 4 and 6 above.

The **Means** of a proportion are its second and third terms; as 3 and 8 above.

The **Antecedents** and **Consequents** of a proportion are the antecedents and consequents of its two ratios.

E.g., 4 and 8 are the antecedents, and 3 and 6 the consequents of the proportion $4:3=8:6$.

Principles of Proportions.

8. *In any proportion the product of the extremes is equal to the product of the means.*

If $a:b=c:d$, we are to prove $ad=bc$.

By Art. 1, $\frac{a}{b}=\frac{c}{d}.$

Clearing of fractions, $ad=bc.$

9. *If the product of two numbers be equal to the product of two other numbers, the four numbers are in proportion.*

Let $ad=bc.$

Dividing by bd, $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$; (1)

by cd, $\frac{a}{c}=\frac{b}{d}$, or $a:c=b:d$; (2)

by ab, $\frac{d}{b}=\frac{c}{a}$, or $d:b=c:a$; (3)

by ac, $\frac{d}{c}=\frac{b}{a}$, or $d:c=b:a$. (4)

Interchanging the ratios in (1), (2), (3), (4),

$c:d=a:b$; (5)

$b:d=a:c$; (6)

$c:a=d:b$; (7)

$b:a=d:c$. (8)

Notice that the two numbers of either product may be taken as the extremes, the other two as the means. In (1) to (4), a and d are the extremes, c and b the means; in (5) to (8), d and a are the means, c and b the extremes.

10. In Art. 9, we may regard the proportions (2) to (8) as being derived from (1), and thus obtain the following properties of a proportion:

(i.) *The means may be interchanged;* as in (2).

(ii.) *The extremes may be interchanged;* as in (3).

(iii.) *The means may be interchanged, and at the same time the extremes;* as in (4).

(iv.) *The means may be taken as the extremes, and the extremes as the means;* as (8) from (1), (7) from (2), etc.

11. *If any three terms of a proportion be given, the remaining term can be found.*

Ex. What is the second term of a proportion, whose first, third, and fourth terms are 10, 16, and 8 respectively?

Letting x stand for the second term, we have

$$10 : x = 16 : 8, \text{ or } 16x = 80; \text{ whence } x = 5.$$

12. *The products, or the quotients, of the corresponding terms of two proportions form again a proportion.*

If $$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}, \tag{1}$$

and $$x : y = z : u, \text{ or } \frac{x}{y} = \frac{z}{u}, \tag{2}$$

we have, multiplying corresponding members of (1) and (2),

$$\frac{ax}{by} = \frac{cz}{du}; \text{ whence } ax : by = cz : du.$$

Dividing the members of (1) by the corresponding members of (2), we have

$$\frac{\frac{a}{x}}{\frac{b}{y}} = \frac{\frac{c}{z}}{\frac{d}{u}}; \text{ whence } \frac{a}{x} : \frac{b}{y} = \frac{c}{z} : \frac{d}{u}.$$

13. *In any proportion, the sum of the first two terms is to the first (or the second) term as the sum of the last two terms is to the third (or the fourth) term.*

Let $$a : b = c : d.$$

Then $$\frac{a}{b} = \frac{c}{d}.$$

Adding 1 to both members, $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

Whence $$a+b:b=c+d:d.$$

In like manner it can be proved that

$$a+b:a=c+d:c.$$

These two proportions are said to be derived from the given proportion by **Composition.**

14. *In any proportion, the difference of the first two terms is to the first (or the second) term as the difference of the last two terms is to the third (or the fourth) term.*

If $$a:b=c:d,$$

then $$a-b:a=c-d:c, \text{ and } a-b:b=c-d:d.$$

The proof is similar to that of Art. 13.

These two proportions are said to be derived from the given proportion by **Division.**

15. *In any proportion, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let $$a:b=c:d.$$

By Art. 13, $$a+b:b=c+d:d;$$

and by Art. 14, $$a-b:b=c-d:d.$$

Then by Art. 12, $$\frac{a+b}{a-b}:1=\frac{c+d}{c-d}:1,$$

or $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Whence $$a+b:a-b=c+d:c-d.$$

This proportion is said to be derived from the given one by **Composition and Division.**

16. A **Continued Proportion** is one in which the consequent of each ratio is the antecedent of the following ratio; as,

$$a:b=b:c=c:d=\text{etc.}$$

17. In the continued proportion

$$a : b = b : c,$$

b is called a **Mean Proportional** between a and c, and c is called the **Third Proportional** to a and b.

18. *The mean proportional between any two numbers is equal to the square root of their product.*

From $$a : b = b : c,$$

we have, by Art. 8, $b^2 = ac$; whence $b = \sqrt{(ac)}$.

19. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $$n_1 : d_1 = n_2 : d_2 = n_3 \quad d_3 = \cdots = v,$$

or $$\frac{n_1}{d_1} = v,\ \frac{n_2}{d_2} = v,\ \frac{n_3}{d_3} = v,\ \cdots.$$

Then, $$n_1 = vd_1,\ n_2 = vd_2,\ n_3 = vd_3,\ \cdots.$$

Adding corresponding members of these equations, we have

$$n_1 + n_2 + n_3 + \cdots = vd_1 + vd_2 + vd_3 + \cdots$$
$$= v\,(d_1 + d_2 + d_3 + \cdots).$$

Therefore $$\frac{n_1 + n_2 + n_3 + \cdots}{d_1 + d_2 + d_3 + \cdots} = v = \frac{n_1}{d_1} = \frac{n_2}{d_2} = \cdots.$$

E.g., $$\frac{1}{2} = \frac{4}{8} = \frac{5}{10} = \frac{1 + 4 + 5}{2 + 8 + 10} = \frac{10}{20}.$$

20. The following examples are applications of the preceding theory:

Ex. 1. Find a mean proportional between 5 and 20.

Let x stand for the required proportional.

Then, by Art. 18, $x = \sqrt{(5 \times 20)} = \pm 10$.

Ex. 2. If $$a : b = c : d,$$

then $$ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2$$

Let $$\frac{a}{b} = \frac{c}{d} = x.$$

Then $$a = bx \text{ and } c = dx.$$

Therefore $$ab + cd = b^2x + d^2x,$$

and $$ab - cd = b^2x - d^2x.$$

We then have $$\frac{ab + cd}{ab - cd} = \frac{b^2x + d^2x}{b^2x - d^2x} = \frac{b^2 + d^2}{b^2 - d^2}.$$

Whence $$ab + cd : ab - cd = b^2 + d^2 : b^2 - d^2.$$

Ex. 3. Solve the equation

$$\frac{\sqrt{(2+x)} + \sqrt{(2-x)}}{\sqrt{(2+x)} - \sqrt{(2-x)}} = 2, = \frac{2}{1}.$$

By composition and division,

$$\frac{\sqrt{(2+x)}}{\sqrt{(2-x)}} = \frac{3}{1}.$$

Squaring and clearing of fractions,

$$2 + x = 18 - 9x; \text{ whence } x = \tfrac{8}{5}.$$

EXERCISES II.

Verify each of the following proportions:

1. $2\frac{1}{2} : 1\frac{1}{3} = 1\frac{1}{2} : \frac{4}{5}$.

2. $14\frac{2}{3} : 4\frac{2}{5} = 200 : 60$.

3. $$\frac{4ab}{a^2 - b^2} : \frac{a^2 + b^2}{a - b} = \frac{2ab}{a^4 - b^4} : \frac{1}{2a - 2b}.$$

Form proportions from each of the following products, in eight different ways:

4. $2x = 3y$.

5. $m^2 = n^2$.

6. $a^3 - b^3 = x^2 - y^2$.

Find a fourth proportional to

7. 1, 2, and 8.

8. $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$.

9. ab, ac, and b.

Find a third proportional to

10. 2 and 6.

11. $\frac{1}{3}$ and $\frac{1}{6}$.

12. a and b.

Find a mean proportional between

13. 2 and 18.

14. $\frac{1}{3}$ and $\frac{3}{4}$.

15. a^2b and ab^2.

16. $\dfrac{a+b}{a-b}$ and $\dfrac{a^2 - b^2}{a^2b^2}$.

17. $\dfrac{a^2 + 1}{a^2 - 1}$ and $\frac{1}{4}(a^4 - 1)$.

Find the value of x to satisfy each of the following proportions:

18. $x : 2 = 12 : 3$. **19.** $161 : 253 = x : 407$. **20.** $7\frac{1}{8} : 1\frac{4}{7} = \frac{7}{9} : x$.

21. $\frac{1}{2} + \sqrt{a} : \frac{1}{4} - a = x : \sqrt{a} - 2a$.

22. $a + b + \dfrac{2b^2}{a-b} : \dfrac{(a+b)^2}{2ab} - 1 = x : a - b$.

Solve each of the following equations:

23. $\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}} = 3$. **24.** $\dfrac{\sqrt{(a+x)} - \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}} = \dfrac{1}{\sqrt{b}}$.

25. $\dfrac{\sqrt{(ax)} + b}{\sqrt{(ax)} - b} = \dfrac{a+b}{a-b}$. **26.** $\dfrac{\sqrt{a} + \sqrt{(bx)}}{a+b} = \dfrac{\sqrt{a} - \sqrt{(bx)}}{a-b}$.

27. $\dfrac{5x+6}{5x-7} = \dfrac{7x+4}{7x-9}$. **28.** $\dfrac{x^2 - x + 6}{x^2 + x - 6} = \dfrac{x^2 + 2x - 3}{x^2 - 2x + 3}$.

29. $\dfrac{x^2 - 5x + 4}{4x - 4} = \dfrac{x^2 - 3x + 2}{3x - 2}$. **30.** $\dfrac{x^2 - 4x + 3}{4x - 3} = \dfrac{x^2 - 6x + 7}{6x - 7}$.

31. Find two numbers whose ratio is $7 : 5$, and the difference of whose squares is 96.

32. A works 6 days with 2 horses, and B works 5 days with 3 horses. What is the ratio of A's work to B's work?

33. The ratio of a father's age to his son's age is $9 : 5$. If the father is 28 years older than the son, how old is each?

34. Find three numbers in a continued proportion whose sum is 39, and whose product is 729.

35. Find two numbers such that if one be added to the first and 8 to the second, the sums will be in the ratio $1 : 2$, and if 1 be subtracted from each number, the remainders will be in the ratio $2 : 3$.

36. What is the ratio of the numerator of a fraction to its denominator, if the fraction be unchanged when a is added to its numerator and b to its denominator?

37. The sum of the means of a proportion is 7, the sum of the extremes is 8, and the sum of the squares of all the terms is 65. What is the proportion?

If $a : b = c : d$, prove that

38. $a + c : b + d = a^2d : b^2c.$

39. $a^2 + b^2 : a^2 - b^2 = c^2 + d^2 : c^2 - d^2.$

40. $(a \pm b)^2 : ab = (c \pm d)^2 : cd.$

41. $2a + 3b : 4a + 5b = 2c + 3d : 4c + 5d.$

42. $a + b : c + d = \sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)}.$

43. $\sqrt{(a^2 + b^2)} : \sqrt{(c^2 + d^2)} = \sqrt[3]{(a^3 + b^3)} : \sqrt[3]{(c^3 + d^3)} = a : c.$

VARIATION.

21. Frequently two numbers or quantities are so related to each other that a change in the value of one produces a corresponding change in the value of the other.

Thus, the distance a train runs in one hour depends upon its speed, and increases or decreases when its speed increases or decreases.

The illumination made by a light depends upon the intensity of the light, and varies when the intensity varies.

The value of y given by the equation $y = 2x - 3$ depends upon the value of x, and varies when the value of x varies.

Thus, if $x = 1$, $y = -1$; if $x = 2$, $y = 1$, etc.

We shall in this chapter consider only the simplest kinds of variation.

22. Direct Variation. — Two quantities are said to *vary directly* one as the other, when their ratio is constant.

Thus, if x varies directly as y, then $\frac{x}{y} = k$, a constant.

For example, if a train runs at a uniform speed, the number of miles it runs varies directly as the number of hours. If it runs at the rate of 30 miles an hour, in 1 hour it will run 30 miles, in 2 hours 60 miles, in 3 hours 90 miles, and so on; and the ratios $1:30$, $2:60$, $3:90$, etc., are equal.

The symbol of direct variation, $\propto$, is read *varies directly as.*

The word *directly* is frequently omitted.

If $y = 3x$, then $y \propto x$ (read *y varies as x*), since $\frac{y}{x} = 3$, a constant.

23. Inverse Variation. — One quantity is said to *vary inversely* as another when the first varies as the *reciprocal* of the second.

Thus, if x varies inversely as y, then $x \propto \frac{1}{y}$.

Therefore, $\frac{x}{\frac{1}{y}} = k$, a constant; whence $xy = k$.

That is, if one quantity varies inversely as another, the product of the quantities is constant.

If 6 men can do a piece of work in 12 hours, 3 men can do the same work in 24 hours, and 1 man in 72 hours, and the products 6×12, 3×24, 1×72 are equal. That is, the number of hours varies inversely as the number of men working.

If $y = \frac{3}{x}$, y varies inversely as x, since $xy = 3$.

24. Joint Variation. — One quantity is said to *vary* as two others *jointly*, when it varies as the product of the others.

Thus, if x varies as y and z jointly, then $\frac{x}{yz} = k$, a constant.

For example, the number of miles a train runs varies as the number of hours and the number of miles it runs an hour jointly. It will run 40 miles in 2 hours at a rate of 20 miles an hour, 90 miles in 3 hours at the rate of 30 miles an hour,

and $$\frac{40}{2 \times 20} = \frac{90}{3 \times 30} = \frac{120}{5 \times 24}.$$

25. One quantity is said to vary directly as a second and inversely as a third, when it varies as the second and the reciprocal of the third jointly.

Thus, if x varies directly as y and inversely as z, then

$\frac{x}{y \cdot \frac{1}{z}} = k$, a constant; or $\frac{xz}{y} = k$.

26. In all the preceding cases of variation, the constant can be determined when any set of corresponding values of the quantities is known.

Ex. 1. If $x \propto y$, and $x = 3$ when $y = 5$, what is the value of the constant?

We have $\frac{x}{y} = k$, or $x = ky$.

Therefore, when $x = 3$ and $y = 5$,

$3 = 5k$, whence $k = \frac{3}{5}$.

Consequently $x = \frac{3}{5}y$.

EXERCISES III.

1. If $x \propto y$, and $x = 10$ when $y = 5$, what is the value of x when $y = 12\frac{1}{2}$?

2. If $x \propto y$, and $x = a$ when $y = \frac{3}{4}a^2$, what is the value of y when $x = a^2b$?

3. If $x \propto y^2$, and $x = 5$ when $y = -3$, what is the value of x when $y = 15$?

4. If $x \propto \sqrt{y}$, and $x = a + m$ when $y = (a - m)^2$, what is the value of x when $y = (a + m)^4$?

5. If $x \propto \frac{1}{y}$, and $x = 3$ when $y = \frac{3}{8}$, what is the value of x when $y = 4\frac{1}{7}$?

6. If $x \propto \frac{y}{z}$, and $x = 4$ when $y = 6$ and $z = 3$, what is the value of x when $y = 5$, and $z = 2$?

7. The circumference of a circle whose radius is 6 feet is 37.7 feet. What is the circumference of a circle whose radius is 9.5 feet, if the circumference varies as the radius?

8. An ox is tied by a rope 20 yards long in the centre of a field, and eats all the grass within his reach in $2\frac{1}{2}$ days. How many days would it have taken the ox to eat all the grass within his reach if the rope had been 10 yards longer? The area varies as the square of the radius.

9. The volume of a sphere whose radius is 7 inches is 1437.3 cubic inches. What is the volume of a sphere whose radius is 10 inches, if it be known that the volume varies as the cube of the radius?

It has been found by experiment that the distance a body falls from rest varies as the square of the time.

10. If a body falls 256 feet in 4 seconds, how far will it fall in 10 seconds?

11. From what height must a body fall to reach the earth after 15 seconds?

It has been found by experiment that the velocity acquired by a body falling from rest varies as the time.

12. If the velocity of a falling body is 160 feet after 5 seconds, what will be the velocity after 8 seconds?

13. How long must a body have been falling to have acquired a velocity of 384 feet?

14. The surface of a cube whose edge is 5 inches is 150 square inches. What is the surface of a cube whose edge is 9 inches, if it be known that the surface varies as the square of its edge?

15. It has been found by experiment that the weight of a body, above the surface of the earth, varies inversely as the square of its distance from the centre of the earth. If a body weighs 30 pounds on the surface of the earth (approximately 4000 miles from the centre), what would be its weight at a distance of 24,000 miles from the surface of the earth?

It has been found by experiment that the illumination of an object varies inversely as the square of its distance from the source of light.

16. If the illumination of an object at a distance of 10 feet from a source of light is 2, what is the illumination at a distance of 40 feet?

17. To what distance must an object which is now 10 feet from a source of light be removed in order that it shall receive only one-half as much light?

18. At what distance will a light of intensity 10 give the same illumination as a light of intensity 8 gives at a distance of 50 feet?

F. Frencik

CHAPTER XXI.

PROGRESSIONS.

1. A **Series** is a succession of numbers, each formed according to some definite law. The single numbers are called the **Terms** of the series.

E.g., in the series

$$1 + 3 + 5 + 7 + 9 + \cdots \qquad (1)$$

each term after the first is formed by adding 2 to the preceding term.

In the series $$1 + 2 + 4 + 8 + \cdots \qquad (2)$$

each term after the first is formed by multiplying the preceding term by 2.

2. The number of terms in a series may be either *limited* or *unlimited.*

A **Finite** series is one of a *limited* number of terms.

An **Infinite** series is one of an *unlimited* number of terms.

In this chapter a few simple and yet very important series will be discussed.

ARITHMETICAL PROGRESSION.

3. An **Arithmetical Series,** or, as it is more commonly called, an **Arithmetical Progression** (A. P.), is a series in which each term, after the first, is formed by adding a constant number to the preceding term. See Art. 1, (1).

4. Evidently this definition is equivalent to the statement, that the difference between any two consecutive terms is constant.

E.g., in the series

$$1 + 3 + 5 + 7 + \cdots$$

we have $$3 - 1 = 5 - 3 = 7 - 5 = \cdots$$

For this reason the constant number of the first definition is called the **Common Difference** of the series.

5. Let a_1 stand for the first term of the series,
a_n for the nth (*any*) term of the series,
d for the common difference,
and S_n for the sum of n terms of the series.

The five numbers a_1, a_n, d, n, S_n are called the **Elements** of the progression.

6. The common difference may be either positive or negative.

If d be *positive*, each term is greater than the preceding, and the series is called a *rising*, or an *increasing* progression.

E.g., $1 + 2 + 3 + 4 + \cdots$, wherein $d = 1$.

If d be negative, each term is less than the preceding, and the series is called a *falling*, or a *decreasing* progression.

E.g., $1 - 1 - 3 - 5 - \cdots$, wherein $d = -2$.

The *n*th Term of an Arithmetical Progression.

7. By the definition of an arithmetical progression,

$$a_1 = a_1,\ a_2 = a_1 + d,\ a_3 = a_2 + d = a_1 + 2d, \text{ etc.}$$

The law expressed by the formulæ for these first three terms is evidently general, and since the coefficient of d in each is one less than the number of the corresponding term, we have

$$\boldsymbol{a_n = a_1 + (n - 1)d.} \qquad \text{(I.)}$$

That is, to find the nth term of an arithmetical progression: *Multiply the common difference by $n - 1$, and add the product to the first term.*

8. Ex. 1. Find the 15th term of the progression, —

$$1 + 3 + 5 + 7 + \cdots$$

We have $a_1 = 1,\ d = 2,\ n = 15;$

therefore $a_{15} = 1 + (15 - 1)2 = 1 + 28 = 29.$

This formula may be used not only to find a_n, when a_1, d, and n are given, but also to find any one of the four numbers involved when the other three are given.

Ex. 2. If $a_5 = 3 (n = 5)$, and $a_1 = 1$, we have $3 = 1 + 4d$; whence $d = \frac{1}{2}$.

The Sum of *n* Terms of an Arithmetical Progression.

9. The successive terms in an arithmetical progression, from the first to the nth inclusive, may be obtained either by repeated additions of the common difference beginning with the first term, or by repeated subtractions of the common difference beginning with the nth term. We may therefore express the sum of n terms in two equivalent ways:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + \overline{n-2} \cdot d) + (a_1 + \overline{n-1} \cdot d),$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - \overline{n-2} \cdot d) + (a_n - \overline{n-1} \cdot d).$$

Whence, by addition,

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n),$$

wherein there are n binomials, $a_1 + a_n$.

Therefore, $2S_n = n(a_1 + a_n)$, or $\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n)}$. (II.)

10. If the value of a_n, given in (I.), be substituted for a_n in (II.), we obtain

$$\boldsymbol{S_n = \frac{n}{2}[2a_1 + (n-1)d]}. \qquad \text{(III.)}$$

Formula (II.) is used when a_1, a_n, and n are given; and (III.) when a_1, d, and n are given.

11. Ex. 1. If $a_1 = 1$, $a_5 = 3$, then $S_5 = \frac{5}{2}(1 + 3) = 10$.

Ex. 2. If $a_1 = -4$, $d = 2$, $n = 12$,

then $S_{12} = \frac{12}{2}[2(-4) + 11 \times 2] = 84$.

Either (II.) or (III.) can be used to determine any one of the five elements a_1, a_n, d, n, S_n, when the three others involved in the formula are known.

Ex. 3. Given $a_1 = -3$, $d = 2$, $S_n = 12$, to find n.

From (III.), $12 = \frac{n}{2}[-6 + 2(n-1)]$,

or $n^2 - 4n = 12$; whence $n = 6$ and -2.

The result 6 gives the series $-3-1+1+3+5+7, =12$.

Since the number of terms must be positive, the negative result, -2, is not admissible. But its meaning may be assumed to be that two terms, beginning with the last and counting toward the first, are to be taken.

12. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five numbers a_1, a_n, d, S_n, n when the three others are given

Ex. Given $d=-2$, $a_n=-16$, $S_n=-60$, to find a_1 and n.

From (I.), $$-16=a_1-2(n-1), \tag{1}$$

and from (II.), $$-60=\frac{n}{2}(a_1-16). \tag{2}$$

Solving (1) and (2), we obtain $n=12$, $a_1=6$; and $n=5$, $a_1=-8$.

The two series are:

$$6+4+2+0-2-4-6-8-10-12-14-16,$$

and $$-8-10-12-14-16,$$

both of which have $d=-2$, $a_n=-16$, $S_n=-60$.

Notice that in this example the sum of the terms which are not common to the two series is 0.

EXERCISES I.

Find the last term, and the sum of the terms, of each of the following arithmetical progressions:

1. $2+6+\cdots$ to 10 terms.
2. $3+1-\cdots$ to 13 terms.
3. $-5-2+\cdots$ to 21 terms.
4. $3+1\frac{1}{2}+\cdots$ to 40 terms
5. $4+1\frac{3}{4}-\cdots$ to 31 terms.
6. $9+11+\cdots$ to n terms.
7. $n+2n+\cdots$ to 16 terms, to m terms.
8. $a+(a+b)+\cdots$ to 20 terms, to n terms.
9. $(m+2)+(4m+5)+\cdots$ to 40 terms, to n terms.
10. $\dfrac{a-1}{a}+\dfrac{a-3}{a}+\cdots$ to 30 terms, to n terms.

In each of the following arithmetical progressions find the values of the two elements not given:

11. $a_1 = 4,\ d = 5,\ n = 10.$ **12.** $a_n = 16,\ d = 2,\ n = 9.$

13. $a_1 = 2\frac{3}{5},\ n = 5,\ a_n = -1.9.$ **14.** $d = -4.8,\ n = 3,\ S_n = 28.5.$

15. $a_n = 13,\ n = 8,\ S_n = 100.$ **16.** $a_n = 2\frac{1}{6},\ n = 12,\ S_n = -7.$

17. $a_1 = 9,\ d = -1,\ a_n = 6.$ **18.** $a_1 = 22\frac{1}{3},\ a_n = -19\frac{2}{3},\ S_n = 20.$

19. $a_1 = 2,\ d = 5,\ S_n = 245.$ **20.** $a_n = 56,\ d = 5,\ S_n = 324.$

Arithmetical Means.

13. The **Arithmetical Mean** between two numbers is a third number, in value between the two, which forms with them an arithmetical progression.

E.g., 2 is an arithmetical mean between 1 and 3.

Let A stand for the arithmetical mean between a and b; then, by the definition of an arithmetical progression,

$$A - a = b - A,$$

whence
$$A = \frac{a + b}{2}.$$

That is, the arithmetical mean between two numbers is half their sum.

14. Arithmetical Means between two numbers are numbers, in value between the two, which form with them an arithmetical progression.

E.g., 2, 3, and 4 are three arithmetical means between 1 and 5.

Ex. Insert four arithmetical means between -2 and 9.

We have $$n = 6,\ a_1 = -2,\ a_6 = 9.$$

From (I.), $$9 = -2 + 5d, \text{ whence } d = \tfrac{11}{5}.$$

The required means are $\frac{1}{5},\ \frac{12}{5},\ \frac{23}{5},\ \frac{34}{5}.$

EXERCISES II.

Insert an arithmetical mean between

1. 45 and 31. **2.** $17\frac{1}{2}$ and $14\frac{1}{2}$. **3.** $2a$ and $-2b$.

4. $\dfrac{a-b}{a+b}$ and $\dfrac{a+b}{a-b}$. **5.** $\dfrac{x+1}{x-1}$ and $-\dfrac{x^3+1}{x^3-1}$.

6. Insert six arithmetical means between 7 and 35.

7. Insert twelve arithmetical means between 37 and -28.

8. Insert nine arithmetical means between $\frac{1}{5}$ and 12.

9. Insert twenty arithmetical means between -16 and 26.

10. Insert six arithmetical means between $a+b$ and $8a-13b$.

Problems.

15. Pr. Find the sum of all the numbers of three digits which are multiples of 7.

The numbers of three digits which are multiples of 7 are

$$7 \times 15,\ 7 \times 16,\ 7 \times 17,\ \cdots,\ 7 \times 142.$$

Their sum is $\qquad 7(15 + 16 + \cdots + 142).$

The series within the parentheses is an arithmetical progression, in which $a_1 = 15$, $d = 1$, $n = 128$, and $a_{128} = 142$.

Therefore $\qquad S_{128} = 10048.$

The required sum is therefore $7 \times 10048, = 70336$.

16. In many examples the elements necessary for determining the required element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. 1. The sixth term of an A. P. is 17, and the eleventh term is 32. Find the first term and the common difference.

We have $\qquad a_6 = 17,\ a_{11} = 32.$

From (I.), $\quad 17 = a_1 + 5d$, and $32 = a_1 + 10d$.

Solving these equations, $a_1 = 2$, $d = 3$.

Or we could have regarded 17 as the first term and 32 as the last term of a progression of six terms. Then, by (I.), $32 = 17 + 5d$, whence $d = 3$.

By (I.) again, $17 = a_1 + 5 \times 3$; whence $a_1 = 2$, as above.

EXERCISES III.

1. Find the sixth term, and the sum of eleven terms, of an A. P. whose eighth term is 11 and whose fourth term is -1.

2. The sixteenth term of an A. P. is -5, and the forty-first term is 45. What is the first term, and the sum of twenty terms?

3. Find the sum of all the even numbers from 2 to 50 inclusive.

4. Find the sum of thirty consecutive odd numbers, of which the last is 127.

5. The sum of the eighth and fourth terms of an A. P. of twenty terms is 24, and the sum of the fifteenth and nineteenth terms is 68. What are the elements of the progression?

6. The sum of the second and twentieth terms of an A. P. is 10, and their product is $23\frac{47}{64}$. What is the sum of sixteen terms?

7. The sixth term of an A. P. is 30, and the sum of the first thirteen terms is 455. What is the sum of the first thirty terms?

8. What value of x will make the arithmetical mean between $x^{\frac{1}{2}}$ and $x^{\frac{1}{4}}$ equal to 6?

9. Find the sum of all even numbers of two digits.

10. How many consecutive odd numbers beginning with 7 must be taken to give a sum 775?

11. Insert between 0 and 6 a number of arithmetical means so that the sum of the terms of the resulting A. P. shall be 39.

12. Find the number of arithmetical means between 1 and 19, if the first mean is to the last mean as 1 to 7.

13. The sum of the terms of an A. P. of six terms is 66, and the sum of the squares of the terms is 1006. What are the elements of the progression?

14. The sum of the terms of an A. P. of twelve terms is 354, and the sum of the even terms is to the sum of the odd terms as 32 to 27. What is the common difference?

15. How many positive integers of three digits are there which are divisible by 9? Find their sum.

16. Show that the sum of $2n+1$ consecutive integers is divisible by $2n+1$.

17. Prove that if the same number be added to each term of an A. P., the resulting series will be an A. P.

18. Prove that if each term of an A. P. be multiplied by the same number, the resulting series will be an A. P.

19. Prove that if in the equation $y = ax + b$, we substitute c, $c+d$, $c+2d$, $\cdots$, in turn for x, the resulting values of y will form an A. P.

20. A laborer agreed to dig a well on the following conditions: for the first yard he was to receive \$2, for the second \$2.50, for the third \$3, and so on. If he received \$42.50 for his work, how deep was the well?

21. On a certain day the temperature rose $\frac{1}{2}°$ hourly from 5 to 11 A.M., and the average temperature for that period was 8°. What was the temperature at 8 A.M.?

22. Twenty-five trees are planted in a straight line at intervals of 5 feet. To water them, the gardener must bring water for each tree separately from a well which is 10 feet from the first tree and in line with the trees. How far has he walked when he has watered all the trees, beginning with the first?

GEOMETRICAL PROGRESSION.

17. A **Geometrical Series**, or, as it is more commonly called, a **Geometrical Progression** (G. P.), is a series in which each term after the first is formed by multiplying the preceding term by a constant number. See Art. 1, (2).

18. Evidently this definition is equivalent to the statement that the ratio of any term to the preceding is constant.

For this reason the constant multiplier of the first definition is called the **Ratio** of the progression.

19. Let a_1 stand for the first term of the series,
a_n for the nth (*any*) term,
r for the ratio,
and S_n for the sum of n terms.

20. The ratio may be either larger or smaller than 1; in the former case the progression is called a *rising* or *ascending* progression; in the latter a *falling* or *descending* progression.

E.g., $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\cdots$, in which $r=\frac{3}{2}$,

and $\frac{1}{2}-1+2-4+8\cdots$, in which $r=-2$,

are ascending progressions; while

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, in which $r=\frac{1}{2}$,

and $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$, in which $r=-\frac{2}{3}$,

are *descending* progressions.

The nth Term of a Geometrical Progression.

21. By the definition of a geometrical progression

$$a_1=a_1,\ a_2=a_1r,\ a_3=a_2r=a_1r^2,\ a_4=a_3r=a_1r^3, \text{ etc.}$$

The law expressed by the relations for these first four terms is evidently general, and since the exponent of r in each is one less than the number of the corresponding term, we have

$$\boldsymbol{a_n=a_1r^{n-1}}. \tag{I.}$$

That is, to find the nth term of a geometrical progression: *Raise the ratio to a power one less than the number of the term, and multiply the result by the first term.*

Ex. 1. If $a_1=\frac{1}{2}$, $r=3$, $n=5$, then $a_5=\frac{1}{2}\cdot 3^4=\frac{81}{2}$.

This relation may also be used to find not only a_n, when a_1, r, and n are given, but also to find the value of any one of the four numbers when the other three are given.

Ex. 2. If $a_1=4$, $a_6=\frac{1}{8}$, $n=6$, then $\frac{1}{8}=4r^5$, whence $r=\frac{1}{2}$.

The Sum of a Geometrical Progression.

22. We have $S_n=a_1+a_1r+a_1r^2+\cdots+a_1r^{n-2}+a_1r^{n-1}$, (1)

and $rS_n=\quad a_1r+a_1r^2+\cdots+a_1r^{n-2}+a_1r^{n-1}+a_1r^n$. (2)

Consequently, subtracting (2) from (1),

$$S_n(1-r)=a_1-a_1r^n,$$

whence
$$\boldsymbol{S_n=\frac{a_1(1-r^n)}{1-r}=\frac{a_1(r^n-1)}{r-1}}. \tag{II.}$$

Substituting a_n for a_1r^{n-1} in (II.), we have

$$S_n = \frac{a_1 - a_n r}{1 - r} = \frac{a_n r - a_1}{r - 1}. \qquad \text{(III.)}$$

The first forms of (II.) and (III.) are to be used when $r < 1$, the second when $r > 1$.

23. Ex. 1. Given $a_1 = 3$, $r = 2$, $n = 6$, to find S_6.

From (II.), $$S_6 = \frac{3\,(2^6 - 1)}{2 - 1} = 189.$$

Formulæ (II.) and (III.) may be used not only to find S_n when a_1, r, and n, or a_1, a_n, and r are given, but also to find the value of any one of the four numbers when the other three are given.

Ex. 2. Given $S_n = -63\frac{1}{2}$, $a_1 = -\frac{1}{2}$, $a_n = -32$, to find r.

By (III.), $-63\frac{1}{2} = \dfrac{-\frac{1}{2} + 32r}{1 - r}$, whence $r = 2$.

24. Formulæ (I.) and (II.), or (I.) and (III.), may be used simultaneously to determine any two of the five elements, a_1, a_n, r, S_n, n, when the three other elements are given.

Ex. Given $r = 2$, $a_n = 16$, $S_n = 31\frac{1}{2}$, to find a_1 and n.

From (III.), $31\frac{1}{2} = \dfrac{16 \times 2 - a_1}{2 - 1}$, whence $a_1 = \frac{1}{2}$.

From (I.), $16 = \frac{1}{2} \cdot 2^{n-1}$, whence $n = 6$.

EXERCISES IV.

Find the last term and the sum of the terms of each of the following geometrical progressions:

1. $3 + 6 + \cdots$ to six terms.
2. $2 - 4 + \cdots$ to ten terms.
3. $32 - 16 + \cdots$ to seven terms.
4. $1\frac{3}{8} + 2\frac{3}{8} + \cdots$ to six terms.
5. $2 - 2^2 + \cdots$ to eleven terms.
6. $\dfrac{2}{\sqrt{2}} + \dfrac{1}{2} + \cdots$ to n terms.
7. $1 + (1 + a) + \cdots$ to four terms, to n terms.

In each of the following geometrical progressions find the values of the elements not given:

8. $a_1 = 1,\ r = 4,\ n = 5.$ **9.** $a_n = 10,\ r = 2,\ n = 4.$

10. $a_n = 96,\ n = 4,\ S_n = 127.5.$ **11.** $r = 10,\ n = 7,\ S_n = 3{,}333{,}333.$

12. $a_1 = 74\frac{2}{3},\ n = 6,\ a_n = 2\frac{1}{3}.$ **13.** $a_1 = .7,\ r = 10,\ a_n = 700.$

14. $a_1 = 1,\ a_n = 512,\ S_n = 1023.$ **15.** $a_n = 3125,\ r = 5,\ S_n = 3905.$

16. $a_1 = 4,\ r = 3,\ S_n = 118{,}096.$ **17.** $a_1 = 100,\ n = 3,\ S_n = 700.$

25. The Sum of an Infinite Geometrical Progression. — If the number of terms in a geometrical progression is unlimited, the exact value of the sum of the series cannot be obtained. Thus, in the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \text{ without end,}$$

the sum continually increases as more and more terms are included in it.

We have
$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} - \frac{(\frac{1}{2})^n}{\frac{1}{2}}$$
$$= 2 - (\tfrac{1}{2})^{n-1}.$$

And
$$S_1 = 1,\ S_2 = 1\tfrac{1}{2},\ S_3 = 1\tfrac{3}{4},\ S_4 = 1\tfrac{7}{8},\ \cdots$$
$$S_{1000} = 2 - (\tfrac{1}{2})^{999};\ \text{and so on.}$$

We thus see that, although the sum of this series grows larger and larger, it does not increase without limit, but approaches the value 2 more and more nearly as more and more terms are included in the sum. Evidently the sum can be made to differ from 2 by as little as we please, by taking a sufficient number of terms.

We therefore call 2 *the limit of the sum* of the series, or more briefly, the *sum* of the series. The *exact* sum 2, however, can never be obtained.

26. In general, when $r < 1$, the term a_1r^n in the formula

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

decreases as n increases. It can be proved, as in the particular example, that this term can be made as small as we please, by taking n sufficiently great.

Therefore, when $r < 1$, we take

$$S = \frac{a_1}{1 - r}$$

as the sum of the infinite geometrical progression.

This theory can be applied to find the value of a repeating (recurring) decimal.

Ex. Verify that $.\dot{6} = \frac{2}{3}$.

We have $.666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots,$

a geometrical progression whose first term is $\frac{6}{10}$ and whose ratio is $\frac{1}{10}$. Consequently

$$S = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{9} = \frac{2}{3}.$$

EXERCISES V.

Find the sum of the following infinite geometrical progressions:

1. $6 + 4 + \cdots.$ **2.** $60 + 15 + \cdots.$ **3.** $10 - 6 + \cdots.$

4. $\frac{1}{2} + \frac{1}{4} + \cdots.$ **5.** $1 - \frac{1}{3} + \cdots.$ **6.** $5 - \frac{1}{2} + \cdots.$

7. $\frac{3}{2} - \frac{2}{3} + \cdots.$ **8.** $\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} + \cdots.$ **9.** $\sqrt{.2} + \sqrt{\frac{1}{125}} + \cdots.$

10. $1 + x + x^2 + \cdots$, when $x < 1$.

11. $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$, when $x > 1$

Find the value of each of the following repeating decimals

12. $.44 \cdots.$ **13.** $.99 \cdots.$ **14.** $.2727 \cdots.$

15. $.015015 \cdots.$ **16.** $.199 \cdots.$ **17.** $1.0909 \cdots.$

18. $.122323 \cdots.$ **19.** $.201475475 \cdots.$

Verify each of the following identities:

20. $\sqrt{.44 \cdots} = .66 \cdots.$ **21.** $\sqrt{.6944 \cdots} = .833 \cdots.$

Geometrical Means.

27. A **Geometrical Mean** between two numbers is a number, in value between the two, which forms with them a geometrical progression.

E.g., $+2$, or -2, is a geometrical mean between 1 and 4.

Let G be the geometrical mean between a and b.

Then by definition of a geometrical progression,

$$\frac{G}{a}=\frac{b}{G}; \text{ whence } \boldsymbol{G}=\pm\sqrt{(\boldsymbol{ab})}.$$

That is, the geometrical mean between two numbers is the square root of their product.

Ex. Find the geometrical mean between 1 and $\frac{4}{9}$. We have

$$G=\pm\sqrt{(1\times\tfrac{4}{9})}=\pm\tfrac{2}{3}.$$

28. **Geometrical Means** between two numbers are numbers, in value between the two, which form with them a geometrical progression. *E.g.*, 4 and 16 are two geometrical means between 1 and 64; and 2, 4, 8, 16, 32 are five geometrical means between 1 and 64.

Ex. Insert five geometrical means between 1 and 729.

We have $a_1=1,\ n=7,\ a_n=729.$

Therefore $729=r^6$, or $r=\pm 3$.

The required means are:

$$\pm 3,\ 9,\ \pm 27,\ 81,\ \pm 243.$$

EXERCISES VI.

Insert a geometrical mean between

1. 2 and 8.
2. 12 and 3.
3. $\frac{1}{5}$ and $\frac{1}{125}$.
4. $\sqrt{a}$ and $\sqrt{(2a)}$.
5. $75\,m^3$ and $3\,mn^4$.
6. $\dfrac{p}{q}$ and $\dfrac{q}{p}$.
7. $(a-b)^2$ and $(a+b)^2$.
8. $(a^2+1)(a^2-1)^{-1}$ and $\frac{1}{4}(a^4-1)$
9. Insert five geometrical means between 2 and 1458.
10. Insert seven geometrical means between 2 and 512.

11. Insert six geometrical means between 3 and -384.

12. Insert six geometrical means between 5 and -640.

13. Insert nine geometrical means between 1 and $\frac{1024}{59049}$.

Problems.

29. Pr. A farmer agrees to sell 12 sheep on the following terms: he is to receive 2 cents for the first sheep, 4 cents for the second, 8 cents for the third, and so on. How much does he receive for the twelfth sheep, and how much for the 12 sheep, and what is the average price?

We have $a_1 = 2,\ n = 12,\ r = 2.$

Then $a_{12} = 2 \times 2^{11} = 2^{12} = 4096.$

And $$S_{12} = \frac{2(2^{12}-1)}{2-1} = 2 \times 4095 = 8190$$

That is, he receives 4096 cents, or \$40.96, for the twelfth sheep, and 8190 cents, or \$81.90, for the 12 sheep.

The average price is $\frac{81.90}{12}, = \$\,6.82\frac{1}{2}$.

30. In many examples the elements necessary for determining the element or elements directly from (I.)–(III.) are not given, but in their place equivalent data.

Ex. The fifth term of a G. P. is 48, and the eighth term is 384. Find the first term and the ratio.

From (I.), $48 = a_1r^4$, and $384 = a_1r^7$;

whence $r^3 = 8$, or $r = 2$. Therefore $a_1 = 3$.

Or, we could have regarded 48 as the first term and 384 as the last term of a progression of four terms. Then by (I.), $384 = 48\,r^3$, whence $r = 2$ as before.

EXERCISES VII.

1. The first term of a G. P. of six terms is 768, and the last term is one-sixteenth of the fourth term. What is the sum of the six terms of the progression?

2. The first term of a G. P. of ten terms is 3, and the sum of the first three terms is one-eighth of the sum of the next three terms. Find the elements of the progression.

3. The twelfth term of a G. P. is 1536, and the fourth term is 6. What is the ratio, and the sum of the first eleven terms?

4. In a G. P. of eight terms, the sum of the first seven terms is $444\frac{1}{2}$, and is to the sum of the last seven terms as 1 to 2. Find the elements of the progression.

5. The sum of the first four terms of a G. P. is 15, and the sum of the terms from the second to the fifth inclusive is 30. What is the first term, and the ratio?

6. Find the elements of a G. P. of six terms whose first term is 1, and the sum of whose first six terms is 28 times the sum of the first three terms.

7. The sum of the first three terms of a G. P. is 21, and the sum of their squares is 189. What is the first term?

8. The product of the first three terms of a G. P. is 216, and the sum of their cubes is 1971. What is the first term, and the ratio?

9. If the numbers 1, 1, 3, 9 be added to the first four terms of an A. P., respectively, the resulting terms will form a G. P. What is the first term, and the common difference of the A. P.?

10. A G. P. and an A. P. have a common first term 3, the difference between their second terms is 6, and their third terms are equal. What is the ratio of the G. P., and the common difference of the A. P.?

11. Show that, if all the terms of a G. P. be multiplied by the same number, the resulting series will form a G. P.

12. Show that the series whose terms are the reciprocals of the terms of a G. P. is a G. P.

13. Show that the product of the first and last terms of a G. P. is equal to the product of any two terms which are equally distant from the first and last terms respectively.

14. A merchant's investment yields him each year after the first, three times as much as the preceding year. If his investment paid him \$ 9720 in four years, how much did he realize the first year and the fourth year?

15. Given a square whose side is $2a$. The middle points of its adjacent sides are joined by lines forming a second square inscribed in the first. In the same manner a third square is inscribed in the second, a fourth in the third, and so on indefinitely. Find the sum of the perimeters of all the squares.

HARMONICAL PROGRESSION.

31. A **Harmonical Progression** (H. P.) is a series the reciprocals of whose terms form an arithmetical progression.

E.g., $$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$
is a harmonical progression, since
$$1+2+3+4+\cdots$$
is an arithmetical progression.

Consequently to every harmonical progression there corresponds an arithmetical progression, and *vice versa*.

32. Any term of a harmonical progression is obtained by finding the same term of the corresponding arithmetical progression and taking its reciprocal.

Ex. Find the eleventh term of the harmonical progression $4, 2, \frac{4}{3}, \cdots$.

The corresponding arithmetical progression is
$$\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \cdots,$$
and its eleventh term is $\frac{11}{4}$.

Therefore the eleventh term of the given progression is $\frac{4}{11}$.

33. No formula has been derived for the sum of n terms of a harmonical progression.

34. A **Harmonical Mean** between two numbers is a number, in value between the two, which forms with them a harmonical progression.

E.g., $\frac{3}{2}$ is a harmonical mean between $\frac{1}{2}$ and $-\frac{3}{2}$.

Let H stand for the harmonical mean between a and b, then $\frac{1}{H}$ is an arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$. Consequently

$$\frac{1}{H}=\frac{\frac{1}{a}+\frac{1}{b}}{2}, \text{ or } H=\frac{2\,ab}{a+b}.$$

Ex. Insert a harmonical mean between 2 and 5.

We have $$H=\frac{2\times 2\times 5}{2+5}=\frac{20}{7}.$$

35. Harmonical Means between two numbers are numbers, in value between the two, which form with them a harmonical progression.

E.g., $\frac{3}{2}$, 1, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{1}{2}$ are five harmonical means between 3 and $\frac{3}{7}$.

Ex. Insert four harmonical means between 1 and 10.

We have first to insert four arithmetical means between 1 and $\frac{1}{10}$, and obtain

$$\tfrac{41}{50},\ \tfrac{32}{50},\ \tfrac{23}{50},\ \tfrac{14}{50}.$$

The required harmonical means are therefore

$$\tfrac{50}{41},\ \tfrac{50}{32},\ \tfrac{50}{23},\ \tfrac{50}{14}.$$

Problems.

36. Pr. 1. The geometrical mean between two numbers is $\frac{1}{2}$, and the harmonical mean is $\frac{2}{5}$. What are the numbers?

Let x and y represent the two numbers.

Then $$\sqrt{(xy)}=\tfrac{1}{2}, \text{ or } xy=\tfrac{1}{4}; \tag{1}$$

and $$\frac{2\,xy}{x+y}=\frac{2}{5}, \text{ or } 5\,xy=x+y. \tag{2}$$

Solving (1) and (2), we obtain $x=1$, $y=\frac{1}{4}$, and $x=\frac{1}{4}$, $y=1$.

EXERCISES VIII.

Find the last term of each of the following harmonical progressions:

1. $1+\frac{2}{3}+\frac{1}{2}+\cdots$ to 8 terms.
2. $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\cdots$ to 15 terms.
3. $2-2-\frac{2}{3}-\cdots$ to 11 terms.
4. $-8-\frac{8}{9}-\frac{8}{17}-\cdots$ to 16 terms.

5. $\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\cdots$ to 25 terms.

6. $\frac{1}{\sqrt{2}}+\frac{1}{1+\sqrt{2}}+\frac{1}{2+\sqrt{2}}+\cdots$ to 30 terms.

Find the harmonical mean between

7. 2 and 4. 8. -3 and 4. 9. $\frac{1}{7}$ and $\frac{1}{9}$.

10. $\frac{1}{x-1}$ and $-\frac{1}{x+1}$. 11. $\frac{a-b}{a+b}$ and $\frac{a+b}{a-b}$.

12. Insert 5 harmonical means between 5 and $\frac{1}{5}$.

13. Insert 10 harmonical means between 3 and $\frac{1}{4}$.

14. Insert 4 harmonical means between -7 and $\frac{1}{2}$.

15. If b be the harmonical mean between a and c, prove that

$$\frac{a-b}{b-c}=\frac{a}{c}.$$

16. The arithmetical mean between two numbers is 6, and the harmonical mean is $\frac{35}{6}$. What are the numbers?

17. If one number exceeds another by two, and if the arithmetical mean exceeds the harmonical mean by $\frac{1}{10}$, what are the numbers?

18. The seventh term of a harmonical progression is $\frac{1}{15}$, and the twelfth term is $\frac{1}{25}$. What is the twentieth term?

19. The tenth term of a harmonical progression is $\frac{1}{5}$, and the twentieth term is $\frac{1}{10}$. What is the first term?

F. Frenick

CHAPTER XXII.

THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS.

1. The expansions of the powers of a binomial, from the third to the fourth inclusive, were given in Ch. XIII., Arts. 7–8, and the laws governing the expansion of these powers were stated.

As yet, however, we cannot infer that these laws hold for the fifth power without multiplying the expansion of the fourth power by $a+b$; nor for the sixth power without next multiplying the expansion of the fifth power by $a+b$; and so on.

If, however, we prove that, provided the laws hold for any particular power, they hold for the next higher power, we can infer, without further proof, that because the laws hold for the fourth power, they hold also for the fifth; then that because they hold for the fifth, they hold also for the sixth, and so on to any higher power.

2. If the laws (i.)–(vi.) hold for the rth power, we have

$$(a+b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{1 \cdot 2}a^{r-2}b^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3}a^{r-3}b^3 + \cdots.$$

Notice that only the first four terms of the expansion are written. But it is often necessary to write any term (the kth, say) without having written all the preceding terms.

To derive this term, observe that the following laws hold for each term of the expansion:

(i.) *The exponent of b is one less than the number of the term (counting from the left).*

Thus in the first term we have $b^{1-1} = b^0 = 1$; in the second, $b^{2-1} = b$; in the tenth, $b^{10-1} = b^9$; and in the kth term, b^{k-1}.

(ii.) *The exponent of a is equal to the binomial exponent less the exponent of b.*

Thus, in the first term we have $a^{r-0} = a^r$; in the second, a^{r-1}; in the tenth, a^{r-9}; and in the kth term, $a^{r-(k-1)}, = a^{r-k+1}$.

(iii.) *The number of factors (beginning with* 1 *and increasing by* 1) *in the denominator of each coefficient, and the number of factors (beginning with r and decreasing by* 1) *in the numerator of each coefficient, is equal to the exponent of b in that term.*

Thus, in the coefficient of the second term the denominator is 1 and the numerator is r; in that of the third term the denominator is $1 \cdot 2$ and the numerator is $r(r-1)$; in the tenth term the denominator is $1 \cdot 2 \cdots 9$ and the numerator is $r(r-1) \cdots (r-8)$; and in the kth term the denominator is $1 \cdot 2 \cdot 3 \cdots (k-1)$, and the numerator is

$$r(r-1) \cdots [r-(k-2)], \ = r(r-1) \cdots (r-k+2).$$

Therefore the kth term in the expansion of $(a+b)^r$ is

$$\frac{r(r-1)(r-2) \cdots (r-k+2)}{1 \cdot 2 \cdot 3 \cdots (k-1)} a^{r-k+1} b^{k-1}.$$

In like manner, any other term can be written.

Thus, the $(k-1)$th term is

$$\frac{r(r-1)(r-2) \cdots (r-k+3)}{1 \cdot 2 \cdot 3 \cdots (k-2)} a^{r-k+2} b^{k-2}.$$

3. We can now prove that, if the laws (i.)–(vi.) hold for $(a+b)^r$, they also hold for $(a+b)^{r+1}$; that is, if they hold for any power they hold for the next higher power. Assuming, then, that the laws hold for $(a+b)^r$, we have

$$\begin{aligned}(a+b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{1 \cdot 2} a^{r-2}b^2 + \cdots \\ + \frac{r(r-1)(r-2) \cdots (r-k+3)}{1 \cdot 2 \cdot 3 \cdots (k-2)} a^{r-k+2} b^{k-2} \\ + \frac{r(r-1)(r-2) \cdots (r-k+3)(r-k+2)}{1 \cdot 2 \cdot 3 \cdots (k-2)(k-1)} a^{r-k+1} b^{k-1} + \cdots.\end{aligned}$$

The first three terms of the expansion are written, then all terms are omitted, except the $(k-1)$th and the kth.

Multiplying the expansion of $(a+b)^r$ by $(a+b)$, we obtain:

$$\begin{aligned}(a+b)^{r+1} &= a^{r+1} + ra^rb + \frac{r(r-1)}{1\cdot 2}a^{r-1}b^2 + \cdots \\ &+ \frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1} + \cdots \\ &+ a^rb + ra^{r-1}b^2 + \cdots + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}a^{r-k+2}b^{k-1} + \cdots \\ &= a^{r+1} + (r+1)a^rb + \left[\frac{r(r-1)}{1\cdot 2} + r\right]a^{r-1}b^2 + \cdots \\ &+ \left[\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)} + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)}\right]a^{r-k+2}b^{k-1} + \cdots.\end{aligned}$$

But $$\frac{r(r-1)}{1\cdot 2} + r = \frac{r^2 - r + 2r}{1\cdot 2} = \frac{(r+1)r}{1\cdot 2};$$

and $$\begin{aligned}&\frac{r(r-1)\cdots(r-k+2)}{1\cdot 2\cdots(k-1)} + \frac{r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-2)} \\ &= \frac{r(r-1)\cdots(r-k+2) + r(r-1)\cdots(r-k+3)(k-1)}{1\cdot 2\cdots(k-1)} \\ &= \frac{r(r-1)\cdots(r-k+3)(r-k+2+k-1)}{1\cdot 2\cdots(k-1)} \\ &= \frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}(a+b)^{r+1} &= a^{r+1} + (r+1)a^rb + \frac{(r+1)r}{1\cdot 2}a^{r-1}b^2 + \cdots \\ &+ \frac{(r+1)r(r-1)\cdots(r-k+3)}{1\cdot 2\cdots(k-1)}a^{r-k+2}b^{k-1} + \cdots.\end{aligned}$$

The laws (i.)–(vi.) hold for the above expansion of $(a+b)^{r+1}$. We therefore conclude that if the expansion holds for $(a+b)^r$, it also holds for $(a+b)^{r+1}$.

Consequently, since the expansion holds for the fourth power, it holds for the fifth, and so on to any positive integral power.

The method of proof employed in this article is called **Proof by Mathematical Induction.**

4. We may now write the expansion of $(a+b)^n$, wherein n is any positive integer:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots.$$

In particular, if $a=1$, and $b=x$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots.$$

5. The expansion of $(a-b)^n$ can be at once written from that of $(a+b)^n$.

We have $\qquad (a-b)^n = [a+(-b)]^n$

$$= a^n + na^{n-1}(-b) + \frac{n(n-1)}{1\cdot 2}a^{n-2}(-b)^2 + \cdots$$

$$= a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \cdots.$$

Observe that the signs of the terms alternate, + and −, beginning with the first, or that the terms containing *even* powers of b are *positive*, and those containing *odd* powers of b are *negative*.

6. When n is a positive integer, the number of terms in the expansion is limited.

E.g., $$(a+b)^5 = a^5 + 5a^4b + \frac{5\cdot 4}{1\cdot 2}a^3b^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}a^2b^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}ab^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}b^5 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}a^{-1}b^6 + \cdots.$$

The coefficients of the seventh and all succeeding terms contain 0 as a factor. Therefore these terms drop out, and the expansion ends with the sixth term. In general, the expansion of $(a+b)^n$ ends with the $(n+1)$th term. For, the coefficients of the $(n+2)$th and all succeeding terms contain $n-n$, or 0, as a factor.

7. The expansion of $(a+b)^n$ may also be written in descending powers of b.

Thus, $(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1 \cdot 2}b^{n-2}a^2 + \cdots,$

wherein b^n is the last term of the expansion given in Art. 4, n the coefficient of the next to the last term, and so on.

We therefore conclude:

In the expansion of $(a+b)^n$, wherein n is a positive integer, the coefficients of terms equally distant from the beginning and end of the expansion are equal.

8. In Exs. 1–2 which follow, the coefficients are computed by the principle given in Ch. XIII., Art. 7 (v.).

Ex. 1. Expand $(1-2x^2)^5$.

We have
$$\begin{aligned}(1-2x^2)^5 &= 1^5 - 5 \cdot 1^4 \cdot (2x^2) + 10 \cdot 1^3 \cdot (2x^2)^2 \\ &\quad - 10 \cdot 1^2 \cdot (2x^2)^3 + 5 \cdot 1 \cdot (2x^2)^4 - (2x^2)^5 \\ &= 1 - 10x^2 + 40x^4 - 80x^6 + 80x^8 - 32x^{10}.\end{aligned}$$

In expanding a binomial, the coefficients of the terms after the middle term may be at once written by the principle of the preceding article. This remark applies to the expansion before it is reduced, as in Ex. 1.

Ex. 2. Find the first five terms of $(a^{-\frac{1}{2}} + 2b^{-2})^{11}$.

We have
$$\begin{aligned}(a^{-\frac{1}{2}} + 2b^{-2})^{11} &= (a^{-\frac{1}{2}})^{11} + 11(a^{-\frac{1}{2}})^{10}(2b^{-2}) + 55(a^{-\frac{1}{2}})^9(2b^{-2})^2 \\ &\quad + 165(a^{-\frac{1}{2}})^8(2b^{-2})^3 + 330(a^{-\frac{1}{2}})^7(2b^{-2})^4 + \cdots \\ &= a^{-\frac{11}{2}} + 22a^{-5}b^{-2} + 220a^{-\frac{9}{2}}b^{-4} + 1320a^{-4}b^{-6} \\ &\quad + 5280a^{-\frac{7}{2}}b^{-8} + \cdots.\end{aligned}$$

9. Ex. Find the seventh term in $(2x-3y)^{11}$.

In the seventh term the exponent of $-3y(=b)$ is 6; the exponent of $2x(=a)$ is $11-6, =5$. The denominator of the coefficient contains six factors beginning with 1, and the numerator contains six factors beginning with 11. Therefore the seventh term is

$$\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2x)^5(-3y)^6, \;= 10777536\,x^5y^6.$$

If the second term of the binomial is negative, it is better in finding a particular term, to write the binomial in the form $[a+(-b)]$, as in the above example.

EXERCISES.

Write the expansion of each of the following powers:

1. $(a+b)^6$. **2.** $(x-y)^7$. **3.** $(a^2+b^2)^8$.

4. $(x^{-1}+y^3)^4$. **5.** $(a^{\frac{1}{2}}-b^4)^5$. **6.** $(x^{-2}+y^{\frac{1}{2}})^6$.

7. $(x^{\frac{1}{2}}-y^{\frac{2}{3}})^4$. **8.** $(a^{-3}+b^{-1})^5$. **9.** $(m^{-\frac{1}{4}}-n^{\frac{1}{2}})^6$.

10. $\left(x-\frac{a}{x}\right)^5$. **11.** $\left(\frac{a}{b}-\frac{b}{a}\right)^6$. **12.** $\left(x+\frac{1}{x^2}\right)^7$.

13. $(a-5)^6$. **14.** $(2x+3y)^5$. **15.** $(4a^2-\frac{1}{5}b^{-\frac{1}{2}})^4$.

16. $(x^2-\sqrt{y})^4$. **17.** $(2\sqrt{a}-\frac{1}{3}\sqrt{b})^5$. **18.** $(x^3-y^2\sqrt{-3})^6$.

19. $\left(\sqrt{\frac{a}{n}}+\sqrt{\frac{n}{a}}\right)^9$. **20.** $\left(\frac{2}{\sqrt[3]{a^2}}-\frac{a\sqrt{a}}{2}\right)^5$. **21.** $\left(n^2+\frac{2a}{n^{-1}}\right)^6$.

22. $(\sqrt{-2}+2x^{-\frac{2}{7}})^7$. **23.** $(\sqrt[4]{a}+\sqrt[4]{b})^8$. **24.** $(a-\sqrt{-a})^8$.

25. $(ab^{-2}-b^2x)^9$. **26.** $(x^2-\sqrt{-x})^9$. **27.** $(a^2b+b^{-3})^{10}$.

28. $[\sqrt{(x+1)}-\sqrt{(x-1)}]^4$. **29.** $[\sqrt[3]{(a+b)}+\sqrt[3]{(a-b)}]^6$.

Simplify each of the following expressions:

30. $(1+\sqrt{-x})^8+(1-\sqrt{-x})^8$. **31.** $(x+\sqrt{-3})^9-(x-\sqrt{-3})^9$.

Write the expansion of each of the following powers:

32. $(1-x+x^2)^3$. **33.** $(2-3x+x^2)^4$.

34. $(1+a^{\frac{1}{2}}-a^{-2})^3$. **35.** $(1-x\sqrt{2}+x^2\sqrt{3})^4$.

Write the

36. 3d term of $(a+b)^{15}$. **37.** 5th term of $(a-b)^{16}$.

38. 6th term of $(a^{\frac{1}{10}}+b^{\frac{1}{5}})^{15}$. **39.** 7th term of $(a^n-a^{-n})^{14}$.

40. 6th term of $\left(\sqrt[3]{m}-\frac{2x}{\sqrt[3]{m^2}}\right)^{12}$. **41.** 15th term of $\left(a^3+\frac{1}{a}\right)^{20}$.

42. 12th term of $(x-\sqrt{-x})^{20}$. **43.** 9th term of $(\sqrt{x}-ax^{\frac{2}{3}})^{16}$.

44. Write the middle term of $(x\sqrt{x}-1)^4$.

45. Write the middle terms of $(a^{\frac{1}{3}}+x^{\frac{1}{2}})^9$.

J. Frencik

CHAPTER XXIII.

PERMUTATIONS AND COMBINATIONS.

DEFINITIONS.

1. The following examples will illustrate the character of an important class of problems.

Pr. **1.** Write the numbers of two figures each which can be formed from the three figures, 4, 5, 6.

We have 45, 54, 46, 64, 56, 65.

Pr. **2.** What committees of two persons each can be appointed from the three persons, A, B, C?

The committees may consist of A, B; A, C; or B, C.

These problems make clear the difference between groups of things, selected from a given number of things, in which *the order is taken into account*, as in Pr. 1, and in which *the order is not taken into account*, as in Pr. 2.

2. We are thus naturally led to the following definitions:

A Permutation of any number of things is a group of some or all of them, *arranged in a definite order.*

A Combination of any number of things is a group of some or all of them, *without reference to order.*

3. It follows from these definitions that two permutations are different when some or all of the things in them are different, or when their order of arrangement is different; and that two combinations are different only when at least one thing in one is not contained in the other.

Thus, *ab* and *ba* are different permutations, but the same combination.

PERMUTATIONS.

4. The permutations of a, b, c, d are:

1	2	3	4
a	ab	abc	$abcd$
		abd	$abdc$
	ac	acb	$acbd$
		acd	$acdb$
	ad	adb	$adbc$
		adc	$adcb$
b	ba	bac	$bacd$
		bad	$badc$
	bc	bca	$bcad$
		bcd	$bcda$
	bd	bda	$bdac$
		bdc	$bdca$
c	ca	cab	$cabd$
		cad	$cadb$
	cb	cba	$cbad$
		cbd	$cbda$
	cd	cda	$cdab$
		cdb	$cdba$
d	da	dab	$dabc$
		dac	$dacb$
	db	dba	$dbac$
		dbc	$dbca$
	dc	dca	$dcab$
		dcb	$dcba$

The permutations two at a time are formed from those one at a time, by annexing to each of the latter each remaining letter in turn; those three at a time from those two at a time in like manner; and so on. Evidently the permutations thus formed are all different.

Of four things, only four permutations one at a time can be formed. And since, in the permutations two at a time formed from those one at a time, each thing is followed by each remaining thing, none of those two at a time are omitted. For a similar reason, none of those three and four at a time are omitted. Therefore the above representation includes all permutations of the four letters, one, two, three, and four at a time.

5. The number of permutations of n things taken r at a time is denoted by the symbol ${}_nP_r$.

Then from the enumeration of the preceding article, we have

$${}_4P_1 = 4,\ {}_4P_2 = 12,\ {}_4P_3 = 24,\ {}_4P_4 = 24.$$

6. When the number of things is large, the preceding method of enumeration becomes laborious.

The following example illustrates a method of deriving a general formula for ${}_nP_r$.

We have $${}_4P_1 = 4.$$

Each permutation one at a time gives as many permutations two at a time as there are things remaining to annex to it in turn, in this case three.

Therefore $${}_4P_2 = {}_4P_1 \times 3 = 4 \times 3.$$

Each permutation two at a time gives as many permutations three at a time as there are things remaining to annex to it in turn, in this case two.

Therefore $${}_4P_3 = {}_4P_2 \times 2 = 4 \times 3 \times 2.$$

In like manner, $${}_4P_4 = {}_4P_3 = 4 \times 3 \times 2 \times 1.$$

In general, $\boldsymbol{{}_nP_r = n(n-1)(n-2)\cdots(n-r+1)}$,

when the $\boldsymbol{n}$ *things are all different.*

Evidently $${}_nP_1 = n. \tag{1}$$

From each permutation of n things one at a time we obtain, by annexing to it each of the $n-1$ remaining things in turn, $n-1$ permutations two at a time.

Therefore $${}_nP_2 = {}_nP_1(n-1) = n(n-1). \tag{2}$$

Again, from each permutation of n things two at a time we obtain, by annexing to it each of the $n-2$ remaining things in turn, $n-2$ permutations three at a time.

Therefore $${}_nP_3 = {}_nP_2(n-2) = n(n-1)(n-2). \tag{3}$$

In like manner,

$${}_nP_4 = {}_nP_3(n-3) = n(n-1)(n-2)(n-3). \tag{4}$$

The method is evidently general. The number subtracted from n in the last factor in (1)–(4) is *one less than the number of things taken at a time.* Therefore,

$${}_nP_r = n(n-1)(n-2)\cdots[n-(r-1)] = n(n-1)(n-2)\cdots(n-r+1).$$

7. Observe that the number of factors in the formula for ${}_nP_r$ is equal to the number of things taken at a time.

E.g., $${}_8P_5 = 8 \times 7 \times 6 \times 5 \times 4 = 6720.$$

8. If all the things are taken at a time, *i.e.*, if $r = n$, we have

$${}_nP_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots 3 \times 2 \times 1.$$

E.g., $${}_5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

9. The continued product

$$n(n-1)(n-2)\cdots 3 \times 2 \times 1$$

is called **Factorial-*n***, and is denoted by the symbol $\underline{|n}$ or $n!$

Therefore the formula of the preceding article may be written

$${}_nP_n = \underline{|n}.$$

E.g., ${}_7P_7 = \underline{|7}$, or $7!$, $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1.$

10. In many applications the things considered are not all different. We will now derive a formula for the number of permutations of n things, *taken all at a time*, when some of them are alike.

Let p of the n things be alike, and suppose the permutations n at a time to be formed. In any one of these permutations, let the p like things be replaced by p unlike things, different from all the rest. Then by changing the order of these p new things only, we can form $\underline{|p}$ permutations from the one permutation. In like manner, $\underline{|p}$ permutations can be formed from each of the given permutations. Therefore

$${}_nP_n \text{ (all different)} = {}_nP_n \times \underline{|p}\,(p \text{ alike}),$$

or $${}_nP_n(p \text{ alike}) = \frac{{}_nP_n}{\underline{|p}} = \frac{\underline{|n}}{\underline{|p}}.$$

In like manner, it can be proved that

$${}_nP_n(p \text{ alike}, q \text{ alike}, \cdots) = \frac{\underline{|n}}{\underline{|p} \times \underline{|q} \times \cdots}.$$

E.g., $${}_8P_8(3 \text{ alike}) = \frac{\underline{|8}}{\underline{|3}} = 6720.$$

EXERCISES I.

Find the values of

1. ${}_{18}P_4$.
2. ${}_{15}P_3$.
3. ${}_{10}P_{10}$.
4. ${}_{20}P_5$.
5. ${}_{n+1}P_3$.
6. ${}_{2n+1}P_5$.
7. ${}_{n+1}P_{n-1}$.
8. ${}_{n+k}P_k$.

Find the value of n, when

9. ${}_nP_4 = 3\,{}_nP_3$.
10. ${}_nP_6 = 20\,{}_nP_4$.
11. ${}_{n+2}P_4 = 15\,{}_nP_3$.
12. ${}_{n+1}P_4 = 30\,{}_{n-1}P_2$.
13. ${}_{n+4}P_3 = 8\,{}_{n+3}P_2$.
14. ${}_{2n+1}P_4 = 140\,{}_nP_3$.

Find the value of k, when

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15. ${}_{10}P_{k+6} = 3\,{}_{10}P_{k+5}$.
16. ${}_7P_{k+1} = 12\,{}_7P_{k-1}$.
17. ${}_{12}P_k = 20\,{}_{12}P_{k-2}$.

18. How many numbers of 4 figures can be formed with 1, 2, 3, 4, 5, 6, 7?

19. How many numbers of 4 figures can be formed with 0, 1, 2, 3, 4, 5, 6, 7?

20. How many even numbers of 4 figures can be formed with 4, 5, 3, 2?

21. In how many ways can 6 pupils be seated in 10 seats?

22. How many numbers of 5 figures can be formed with 1, 2, 3, 4, 5, 6, 7, 8, 9, if the figure 7 be in the middle of each number?

23. How many permutations can be formed with the letters in the word *Philippine?*

24. How many permutations can be formed with the letters in the word *Iloilo?*

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25. In how many ways can 7 men be seated at a round table?

26. In how many ways can a bracelet be made by stringing together 7 pearls of different shades?

COMBINATIONS.

11. The formula for the number of combinations of n things, r at a time, which is denoted by ${}_nC_r$, is most readily obtained by deriving a relation between ${}_nP_r$ and ${}_nC_r$. The method will be illustrated by a particular example.

The combinations of the four letters a, b, c, d, taken three at a time, evidently are: abc, abd, acd, bcd. From the combination abc we obtain, by changing the order of the letters in all possible ways, $\lfloor 3$ permutations. In like manner, each of the combinations gives $\lfloor 3$ permutations.

Therefore

$${}_4P_3 = {}_4C_3 \times \lfloor 3, \text{ or } {}_4C_3 = \frac{{}_4P_3}{\lfloor 3} = \frac{4 \times 3 \times 2}{\lfloor 3}.$$

In general,

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r},$$

wherein the n things are all different.

For, from each combination that contains r things can be formed $\lfloor r$ permutations, by changing the order of the things in all possible ways. Therefore

$${}_nP_r = {}_nC_r \times \lfloor r, \text{ or } {}_nC_r = \frac{{}_nP_r}{\lfloor r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}.$$

E.g.,
$${}_8C_3 = \frac{8 \times 7 \times 6}{\lfloor 3} = 56.$$

12. The formulæ for ${}_nC_1, {}_nC_2, {}_nC_3, \cdots, {}_nC_r$ may be represented by the following abbreviations:

$$n = \frac{n}{1} = \binom{n}{1},\quad \frac{n(n-1)}{1 \cdot 2} = \binom{n}{2},\quad \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \binom{n}{3},$$

$$\cdots,\quad \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} = \binom{n}{r}.$$

Observe that in the symbolic notation the upper number is the number of things, and the lower number is the number taken at a time.

E.g.,
$${}_7C_4 = \binom{7}{4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35.$$

13. The formula for ${}_nC_r$ can be put in a more convenient form for purposes of theory.

We have

$$ {}_nC_r = \frac{n(n-1)\cdots(n-r+1)\times \cancel{(n-r)(n-r-1)\cdots 3\times 2\times 1}}{\underline{|r}\times \cancel{(n-r)(n-r-1)\cdots 3\times 2\times 1}} $$

$$ = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}. $$

14. We have $$ {}_nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}, $$

and $$ {}_nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}. $$

Therefore, $$ {}_n\boldsymbol{C}_r = {}_n\boldsymbol{C}_{n-r}. $$

That is, *the number of combinations of n dissimilar things r at a time is equal to the number of combinations of the n things n − r at a time.*

This relation is also evident from the definition of a combination. For, every time that r things are taken from the n things to form a combination, there is left a combination of $n-r$ things.

E.g., $$ {}_{100}C_{98} = {}_{100}C_2 = \frac{100\times 99}{1\times 2} = 4950. $$

This relation is thus useful in computing the number of combinations when the number of things taken at a time is large.

15. The Greatest Value of ${}_n\boldsymbol{C}_r$. — We have

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r} $$

$$ = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\times\frac{n-r+1}{r} $$

$$ = {}_nC_{r-1}\times\frac{n-r+1}{r} = {}_nC_{r-1}\left(\frac{n+1}{r}-1\right). \qquad (1) $$

Also, $$ {}_nC_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)}{1\cdot 2\cdot 3\cdots r(r+1)} $$

$$ = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}\times\frac{n-r}{r+1} $$

$$ = {}_nC_r\times\frac{n-r}{r+1}. $$

all a [illegible]

Whence $$ {}_nC_r = {}_nC_{r+1} \times \frac{r+1}{n-r}. \qquad (2)$$

From (1),

$$ {}_nC_r > {}_nC_{r-1}, \text{ when } \frac{n+1}{r} - 1 > 1, \text{ or } r < \frac{n+1}{2}. \qquad (3)$$

That is, the number of combinations of n things, taken any number less than $\frac{n+1}{2}$ at a time, is greater than the number of combinations taken one less at a time, and therefore greater than the number of combinations taken any number less at a time.

From (2), ${}_nC_r > {}_nC_{r+1}$, when $\frac{r+1}{n-r} > 1$, or $r > \frac{n-1}{2}$. (4)

That is, the number of combinations of n things, taken any number greater than $\frac{n-1}{2}$ at a time, is greater than the number of combinations taken one more at a time, and therefore greater than the number of combinations taken any number more at a time. Consequently, ${}_nC_r$ is greatest when r, an integer, lies between $\frac{n-1}{2}$ and $\frac{n+1}{2}$.

(i) n *Even*. Let $n = 2m$. Then r is an integer in value between $\frac{2m-1}{2}$, $= m - \frac{1}{2}$, and $\frac{2m+1}{2}$, $= m + \frac{1}{2}$. That is, $r = m$, $= \frac{n}{2}$. Therefore, when n is even, the greatest number of combinations is ${}_nC_{\frac{n}{2}}$.

(ii) n *Odd*. Let $n = 2m + 1$. Then r should have an integral value between $\frac{2m}{2}$, $= m$, and $\frac{2m+2}{2}$, $= m + 1$. This is evidently impossible, since m and $m+1$ are consecutive integers.

But, when $r = m$, $< m + 1$, $= \frac{n+1}{2}$, then by (3),

$${}_{2m+1}C_m > {}_{2m+1}C_{m-1};$$

$n - r + 1$

and, when $r = m + 1$, $> m$, $= \frac{n-1}{2}$, then by (4),

$${}_{2m+1}C_{m+1} > {}_{2m+1}C_{m+2}.$$

Also, by Art. 14, $\quad {}_{2m+1}C_m = {}_{2m+1}C_{2m+1-m}, = {}_{2m+1}C_{m+1}.$

Consequently, when n is odd, the greatest number of combinations is

$${}_{2m+1}C_m, = {}_{2m+1}C_{m+1}, \text{ or } {}_nC_{\frac{n-1}{2}}, = {}_nC_{\frac{n+1}{2}}.$$

Ex. 1. When $n = 4$, the greatest number of combinations is ${}_4C_2$.

We have $\quad {}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4,\ {}_4C_4 = 1.$

Ex. 2. When $n = 5$, the greatest number of combinations is

$${}_5C_2, = {}_5C_3.$$

We have $\quad {}_5C_1 = 5,\ {}_5C_2 = 10,\ {}_5C_3 = 10,\ {}_5C_4 = 5,\ {}_5C_5 = 1$

EXERCISES II.

Find the values of

1. ${}_{11}C_5$. 2. ${}_{15}C_7$. 3. ${}_{25}C_{20}$. 4. ${}_{98}C_{96}$. 5. ${}_nC_{n-5}$.

Find the value of n, when

6. $2\,{}_nC_5 = 9\,{}_{n-2}C_5$. 7. $3\,{}_nC_3 = 10\,{}_{n-2}C_2$. 8. $4\,{}_{n+1}C_4 = 15\,{}_{n-1}C_3$.
9. ${}_{n+1}P_4 = 112\,{}_{n-1}C_2$. 10. ${}_{n+1}P_4 = 84\,{}_{n-1}C_3$. 11. ${}_nP_2 = 24\,{}_nC_{n-1}$.

Find the value of k, when

12. ${}_8P_k = 24\,{}_8C_k$ 13. ${}_6P_{k+1} = 48\,{}_6C_k$. 14. ${}_{10}P_k = 144\,{}_{10}C_{k-1}$.

15. Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

16. In how many ways can a committee of 4 men be appointed from 25 men?

17. In how many ways can 3 books be selected from 15 books?

18. In a plane are 20 points, no 3 of which are in the same straight line. How many triangles can be formed with 3 points as vertices? How many quadrilaterals, with 4 points as vertices? How many hexagons, with 6 points as vertices?

Find the values of r and ${}_nC_r$, when ${}_nC_r$ is greatest, in

19. ${}_7C_r$. 20. ${}_8C_r$. 21. ${}_{11}C_r$. 22. ${}_{14}C_r$. 23. ${}_{17}C_r$.

TWO IMPORTANT PRINCIPLES.

16. The following example illustrates an important principle.

Pr. Between two cities A and B there are five railroad lines. In how many ways can a man go from A to B and return by a different road?

He can go to B in either of five ways. With each of these five ways he has a choice of four ways of returning. Hence he can make the round trip in $5 \times 4, = 20$, ways.

Evidently, if he were not required to return by a different road he could make the trip in $5 \times 5, = 25$, ways.

The general principle is:

If one thing can be done in ***a*** *ways, and another thing can be done in* ***b*** *ways, and the doing of the first thing does not interfere with the doing of the second, the two things can be done in* ***ab*** *ways.*

The truth of the principle is evident.

17. The following relation will be useful in subsequent work:

$$\begin{aligned} {}_{m+n}C_r = {}_mC_r + {}_mC_{r-1}\,{}_nC_1 + {}_mC_{r-2}\,{}_nC_2 + \cdots + {}_mC_2\,{}_nC_{r-2} \\ + {}_mC_1\,{}_nC_{r-1} + {}_nC_r, \qquad (1) \end{aligned}$$

in which $m >$ or $= r$, $n >$ or $= r$.

The number of combinations of the $m + n$ things r at a time is evidently the sum of:

The number of combinations of m things taken r at a time, or ${}_mC_r$.

The number of combinations of m things taken $r-1$ at a time, multiplied by the number of combinations of n things taken one at a time, or ${}_mC_{r-1}\,{}_nC_1$. And so on.

This relation may be written

$$\binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}. \qquad (2)$$

18. The relation of the preceding article requires m, n, and r to be integers. Evidently, however, the second member of

n - r + 1
r - R + 1

(2) could be made identical with the first member by ordinary reduction. We, therefore, conclude that this relation holds for all rational values of m and n, provided r is a positive integer.

PROBLEMS.

19. Pr. **1.** In how many ways can a committee of 3 Republicans and 4 Democrats be appointed from 18 Republicans and 12 Democrats?

The 3 Republicans can be chosen in ${}_{18}C_3$, $=816$, ways, and the 4 Democrats in ${}_{12}C_4$, $=495$, ways. Since any 3 Republicans can be associated with any 4 Democrats to form the committee, the required number of ways is 816×495, $=403,920$.

Pr. **2.** A box contains 20 balls numbered 1 to 20. In how many ways can 7 balls be selected, if 1 be included, and 2, 3 be excluded?

We first set aside 1 to be included, and 2, 3 to be excluded, and from the remaining 17 balls select 6 balls. Then 1 may be combined with each of the latter in one way, giving a combination of 7 balls. Therefore the problem is equivalent to determining the number of combinations of 17 things, 6 at a time.

Hence $${}_{17}C_6 = \frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, = 12376,$$

is the required number of ways.

EXERCISES III.

1. A man has 3 coats, 4 vests, and 5 pairs of trousers. In how many ways can he dress?

2. In how many ways can 4 white balls, 3 black balls, and 2 red balls be selected from 8 white balls, 7 black balls, and 5 red balls?

3. In how many ways can permutations be formed, with 10 consonants and 4 vowels, each one to contain 5 consonants and 2 vowels?

4. In how many ways can 4 hearts, 3 diamonds, 2 clubs, and 1 spade be drawn from a pack containing 13 cards of each kind?

5. How many numbers of 7 figures can be formed with 1, 2, 3, 4, 5, 6, 7, if the figures 4, 5, 6 be kept together?

6. How many permutations of 10 letters can be formed from 5 consonants and 5 vowels, if no two consonants be adjacent?

7. How many permutations of 9 letters can be formed from 5 consonants and 4 vowels, if each vowel be placed between two consonants?

8. In a school are 96 pupils. In how many ways can a teacher divide them into sections of 12?

9. In how many ways can 4 ladies and 4 gentlemen be seated at a square table, so that a gentleman and a lady shall be seated at each side?

10. How many throws can be made with 2 dice, if such throws as 1, 2 and 2, 1 be regarded as the same? How many with 3 dice?

11. In how many ways can the sum 10 be thrown with 2 dice? With 3 dice?

12. A box contains 15 balls, numbered 1 to 15. In how many ways can 5 balls be selected, if 1, 2, 3 be included? In how many ways, if 1, 2 be included, and 3 excluded? In how many ways, if any two of the numbers 1, 2, 3 be included, the other excluded?

13. In how many ways can 10 different coins be arranged in a row, if the faces of the coins are distinct? In how many ways can they be arranged in a circle?

14. In how many ways can a number of 6 figures be formed with 1, 1, 1, 2, 2, 3, the first and last figure of each number to be an even digit?

15. In how many ways can 7 gentlemen and 10 ladies arrange a game of lawn tennis, each side to consist of 1 lady and 1 gentleman?

CHAPTER XXIV.

VARIABLES AND LIMITS.

VARIABLES.

1. A **Variable** is a number that may have a series of different values in the same investigation or problem.

A **Constant** is a number that has a fixed value in an investigation or problem.

Thus, if d be the number of feet a body has fallen from rest in s seconds, it has been shown by experiment that

$$d = 16\,s^2.$$

As the body falls, the distance d and the time s are variables, and 16 is a constant.

Again, time measured from a past date is a variable, while time measured between two fixed dates is a constant.

2. The constants in a mathematical investigation are, as a rule, general numbers, and are represented by the first letters of the alphabet, a, b, c, etc.; variables are usually represented by the last letters, x, y, z, etc.

3. A variable which has a definite value, or set of values, corresponding to a value of a second variable, is called a **Function** of the latter.

Thus, $16\,x^2$, $\pm\sqrt{(a^2 - x^2)}$, etc., are functions of x; corresponding to any value of x, the first function has one value, the second has two values.

Again, the area of a circle is a function of its radius; the distance a train runs is a function of the time and speed.

4. Much simplicity is introduced into mathematical investigations by employing special symbols for functions.

The symbol $f(x)$, read *function of x*, is very commonly used to denote a function of x.

Thus, $f(x)$ may denote x^2+1 in one investigation, ax^2+bx+c in another.

5. The result of substituting a particular value for the variable in a given expression may be indicated by substituting the same value for the variable in the functional symbol.

Thus, if $f(x) = x^2 + 1$, then $f(a) = a^2 + 1$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0 + 1 = 1$.

EXERCISES I.

1. Given $f(x) = 5x^2 - 3x + 2$; find $f(3), f(0), f(-4), f(x^2)$.
2. Given $f(x) = (x - a)(x - b)(x - c)$; find $f(a), f(b)$.
3. Given $f(x) = x^2 + 1$; find $f(x^2)$, $[f(x)]^2$.
4. Given $f(x) = x^2 - 3x + 2$; find $f(x + 4)$, $f(x + h)$.
5. Given $f(x) = a^x$; find $f(0)$, $f(4)$, $f(-5)$, $f(x^2)$, $f(a)$.
6. Given $f(x) = x^3 + px^2 + qx + r$; find $f\left(y - \frac{p}{3}\right)$.
7. Given $f(m) = 1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 + \cdots$;

   find $f(5)$, $f(\frac{2}{3})$, $f(-3)$, $f(0)$.

LIMITS.

6. When the difference between a variable and a constant may become and remain less than any assigned positive number, however small, the constant is called the **Limit** of the variable.

Let the point P move from A towards B (Fig. 1) in the following way: First to P_1, one-half of the distance from A to B; next from P_1 to P_2, one-half of the distance from P_1 to B;

then from P_2 to P_3, one-half of the distance from P_2 to B; and so on.

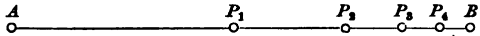


FIG. 1.

Evidently, as P thus moves from A to B, its variable distance from A becomes more and more nearly equal to AB, and the difference between AP and AB can be made less than any assigned distance, however small, by continuing indefinitely the motion of P. Therefore AB is the limit of the variable AP

If we call the distance from A to B unity, we have

$$AP_1 = \tfrac{1}{2},\ P_1P_2 = \tfrac{1}{4},\ P_2P_3 = \tfrac{1}{8},\ P_3P_4 = \tfrac{1}{16},\ \cdots.$$

Hence,

$$AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$$

But, by Ch. XXI, Art. 25, the variable sum of the series on the right approaches 1 as a limit. That is,

$$\text{limit of } (AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots) = AB.$$

7. It follows from the definition of a limit that the variable may be always greater, or always less, or sometimes greater and sometimes less than its limit.

Thus, by Ch. XXI, Art. 25, we have

$$\text{limit } (1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8} - \cdots) = 0, \qquad (1)$$

$$\text{limit } (1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots) = 2, \qquad (2)$$

$$\text{limit } (1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots) = \tfrac{2}{3}. \qquad (3)$$

And in (1), $S_1 = 1,\ S_2 = \tfrac{1}{2},\ S_3 = \tfrac{1}{4},\ S_4 = \tfrac{1}{8},\ \cdots;$ (4)

in (2), $S_1 = 1,\ S_2 = \tfrac{3}{2},\ S_3 = \tfrac{7}{4},\ S_4 = \tfrac{15}{8},\ \cdots;$ (5)

in (3), $S_1 = 1,\ S_2 = \tfrac{1}{2},\ S_3 = \tfrac{3}{4},\ S_4 = \tfrac{5}{8},\ \cdots.$ (6)

Evidently the variable in each of these examples is the sum, which changes as the number of terms increases.

8. The symbol, $\doteq$, read *approaches as a limit,* or simply *approaches,* is placed between a variable and its limit.

The word limit may be abbreviated to lim.

Thus, $\lim_{x \doteq 1}(1-x)=0$, read *the limit of* $1-x$, *as* x *approaches* 1, *is* 0.

Infinites and Infinitesimals.

9. The following considerations lead to important mathemati cal concepts:

The fractions

$$\frac{2}{.1}, = 20;\ \frac{2}{.01}, = 200;\ \frac{2}{.001}, = 2000;\ \frac{2}{.0001}, = 20000;\ \text{etc.},$$

are particular values of the fraction $\frac{n}{x}$, in which the denominator x is assumed to be a variable. It is evident that the value of this fraction can be made greater than any assigned number, however great, by taking its denominator sufficiently small.

A variable which can become and remain numerically greater than any assigned positive number, however great, is called an **Infinite Number,** or simply an **Infinite.**

An infinite variable is denoted by the symbol ∞.

10. The numbers, variables and constants, which have been hitherto used in this book are, for the sake of distinction, called **Finite Numbers.**

11. The fractions

$$\frac{2}{10}, = .2;\ \frac{2}{100}, = .02;\ \frac{2}{1000}, = .002;\ \frac{2}{10000}, = .0002;\ \text{etc.}$$

are also particular values of the fraction $\frac{n}{x}$, in which, as above, the denominator x is assumed to be a variable. It is evident that the value of the fraction $\frac{n}{x}$ can also be made less than any assigned number, however small, by taking the denominator sufficiently great.

A variable which can become and remain numerically less than any assigned positive number, however small, is called an **Infinitesimal.**

No symbol by which to denote an infinitesimal variable has been generally adopted.

It follows from the definition that the limit of an infinitesimal is 0.

12. It is important to keep in mind that both infinites and infinitesimals are *variables*. Their definitions imply that *fixed* values cannot be assigned to them.

An infinitesimal should therefore not be confused with 0, which is the *constant* difference between any two equal numbers.

13. The statement, *x becomes infinite*, or *x increases numerically beyond any assigned positive number, however great*, is frequently abbreviated by the expression, $x \doteq \infty$.

14. The conclusions reached in Arts. 9 and 11 can now be restated thus:

(i.) *If the numerator of a fraction remain finite and not* 0, *and the denominator approach zero, the value of the fraction will become infinite;* or stated symbolically,

$$\frac{n}{x} \doteq \infty, \text{ as } x \doteq 0,$$

wherein n is finite and not 0.

(ii.) *If the numerator of a fraction remain finite and not* 0, *and the denominator become infinite, the value of the fraction will approach* 0; or stated symbolically,

$$\frac{n}{x} \doteq 0, \text{ as } x \doteq \infty,$$

wherein n is finite and not 0.

Observe that these principles hold not only when n is a constant, not 0, but also when n is a variable, provided it does not become infinite.

15. *The difference between a variable and its limit is evidently an infinitesimal;* that is,

$$\text{if } lim\, x = a, \text{ then } lim\,(x - a) = 0$$

Consequently, if $\lim x = a$, we have

$$x - a = x', \text{ or } x = a + x',$$

wherein x' is a variable whose limit is 0.

Conversely, if $x = a + x'$, and x' be a variable whose limit is 0, then $\lim x = a$.

16. *If the limit of a variable be* 0, *the limit of the product of the variable and any finite number is* 0; that is,

if $\lim x = 0$, and a be any finite number, $\lim ax = 0$.

Let k be any number, however small. Then x can be made less numerically than $\frac{k}{a}$, and, therefore, ax less than k. Hence, $\lim ax = 0$.

Fundamental Principles of Limits.

17. (i.) *If two variables be always equal, and one of them approach a limit, the other approaches the same limit.* That is,

if $x = y$, and $x \doteq a$, then $y \doteq a$.

(ii.) *If two variables be always equal as they approach their limits, their limits are equal.* That is,

if $\lim x = a$, $\lim y = b$, and $x = y$, then $a = b$.

(iii.) *The limit of the algebraical sum of a finite number of variables is the sum of their limits.* That is,

if $\lim x = a$, $\lim y = b$, $\cdots$, then $\lim (x + y + \cdots) = a + b + \cdots$.

(iv.) *The limit of the product of a finite number of variables is the product of their limits, if none of the limits be* ∞. That is,

if $\lim x = a$, $\lim y = b$, $\cdots$, then $\lim (xy \cdots) = ab \cdots$.

(v.) *The limit of the quotient of two variables is the quotient of their limits, if the limit of the divisor be not* 0. That is,

if $\lim x = a$, $\lim y = b$, then $\lim \left(\frac{x}{y}\right) = \frac{a}{b}$, when $\lim y \neq 0$.

The proofs follow:

(i.) We have $x = a + x'$, wherein, by Art. 15, x' is a variable whose limit is 0. Then, since $y = x$ always, we have $y = a + x'$. Hence $\lim y = a$.

(ii.) This principle follows directly from (i.).

(iii.) We have $x = a + x', y = b + y', \cdots$, wherein, by Art. 15, $x', y', \cdots$ are variables whose limits are 0.

Then $x + y + \cdots = (a + b + \cdots) + (x' + y' + \cdots)$.

Let k be any assigned number, however small. Then each of the variables $x', y', \cdots$ can be made less than $\frac{k}{n}$, wherein n is the number of variables. Therefore, $x' + y' + \cdots$ can be made less than k. Consequently $\lim (x + y \cdots) = a + b + \cdots$.

(iv.) We have $x = a + x'$, $y = b + y', \cdots$, wherein $x', y', \cdots$ are variables whose limits are 0, and $a, b, \cdots$ are finite.

Then $xy = ab + bx' + ay' + x'y'$. Therefore, by (iii.),

$$\lim xy + = \lim ab + \lim bx' + \lim ay' + \lim x'y'$$
$$= ab, \text{ since } \lim bx' = 0, \cdots, \text{ by Art. 16.}$$

In like manner, the principle can be proved for any finite number of factors.

(v.) Let $\frac{x}{y} = q$, or $x = yq$. Then, by (iv.), $\lim x = \lim y \lim q$.

Therefore, $\lim q = \frac{\lim x}{\lim y}$, or $\lim \frac{x}{y} = \frac{\lim x}{\lim y}$.

Indeterminate Fractions.

18. It follows from the definition of a fraction that $\frac{0}{0}$ is a number which multiplied by 0 gives 0. But any finite number multiplied by 0 gives 0, or $0\,n = 0$. Consequently $\frac{0}{0}$ may denote *any number whatever*.

For this reason, such a fraction is called an **Indeterminate Fraction.**

19. The fraction $\frac{x^2 - 9}{x - 3}$ becomes $\frac{0}{0}$ when $x = 3$, and has no definite value. But as long as $x \neq 3$, however little it may differ from 3, we may perform the indicated division. We therefore have

$$\frac{x^2 - 9}{x - 3} = x + 3, \text{ when } x \neq 3.$$

Now since the limit of the fraction depends upon values of x which differ from 3, however little, we have

$$\lim_{x \doteq 3} \frac{x^2-9}{x-3} = \lim_{x \doteq 3} (x+3) = 6.$$

Although the given fraction is indeterminate, it is clearly desirable that it shall have a definite value. *We therefore assign to* $\frac{x^2-9}{x-3}$ *the value* 6, *when* $x = 3$.

That is, we *define* an indeterminate fraction to be the limit of the fraction as the variable approaches that value which renders it indeterminate. In this way we may obtain a definite value when the fraction involves but one variable.

20. The fraction $\frac{\infty}{\infty}$ is a number which multiplied by ∞ gives ∞. But any finite number multiplied by ∞ gives ∞. Therefore $\frac{\infty}{\infty}$ is also an *indeterminate fraction.*

21. The fraction $\frac{n-1}{n+1}$ becomes $\frac{\infty}{\infty}$, as $n \doteq \infty$. Dividing numerator and denominator by n, we have

$$\frac{n-1}{n+1} = \frac{1-\frac{1}{n}}{1+\frac{1}{n}}.$$

Since

$$\frac{1}{n} \doteq 0, \text{ as } n \doteq \infty,$$

$$\lim_{n \doteq \infty} \frac{n-1}{n+1} = \lim_{n \doteq \infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 1.$$

EXERCISES II.

Find the limiting values of the following fractions:

1. $\frac{x^2-6x+5}{x^2-8x+15}$, when $x \doteq 5$. 2. $\frac{x^2-3x+2}{x^2+x-6}$, when $x \doteq 2$.

3. $\frac{3a^2-ab-2b^2}{9a^2+12ab+4b^2}$, when $a \doteq -\frac{2}{3}b$.

4. $\dfrac{9x^2-30xy+25y^2}{3x^2-2xy-5y^2}$, when $x \doteq \frac{5}{3}y$.

5. $\dfrac{x^3+2x^2-x-2}{x^2+x-2}$, when $x \doteq 1$.

6. $\dfrac{(x^2-1)^2}{x-1}$, when $x \doteq 1$.

7. $\dfrac{x^3-3x+2}{x^2-7x+5}$, when $x \doteq 1$.

8. $\dfrac{x^2-4x+5}{x^3-3x+2}$, when $x \doteq 1$.

9. $\dfrac{a^{2x}-1}{a^x-1}$, when $x \doteq 0$.

10. $\dfrac{2x+1}{(4x^2-1)^3}$, when $x \doteq -\frac{1}{2}$.

Find the limiting values of the following fractions when $n \doteq \infty$:

11. $\dfrac{n+1}{n^2-1}$.

12. $\dfrac{n^2-9}{n+3}$.

13. $\dfrac{n(n-1)}{\lfloor 2} \cdot \dfrac{1}{n^2}$.

14. $\dfrac{n^2-3n+2}{2n^2-3n+4}$.

15. $\dfrac{nx(nx-1)(nx-2)}{\lfloor 3} \cdot \dfrac{1}{n^3}$.

Indeterminate Solutions.

22. The preceding principles may be further illustrated by examining the infinite and indeterminate solutions of certain problems.

Pr. A merchant buys four pieces of goods. In the second piece there are 3 yards less than in the first, in the third 7 yards less than in the first, and in the fourth 10 yards less than in the first. The number of yards in the first and fourth is equal to the number of yards in the second and third. How many yards are there in the first piece?

Let x stand for the number of yards in the first piece; then the number of yards in the second piece is $x-3$; in the third piece, $x-7$; in the fourth piece, $x-10$. Therefore, by the condition of the problem, we have

$$x+(x-10)=(x-3)+(x-7), \text{ or } 2x-10=2x-10.$$

This equation is an identity, and is therefore satisfied by *any finite value of* x.

If it be solved in the usual way, we obtain

$$(2-2)x = 10-10, \text{ or } x = \frac{10-10}{2-2} = \frac{0}{0}.$$

That is, the conditions of the problem will be satisfied by any number of yards in the first piece.

Infinite Solutions.

23. Pr. A cistern has three pipes. Through the first it can be filled in 24 minutes; through the second in 36 minutes; through the third it can be emptied in a minutes. In what time will the cistern be filled if all the pipes be opened at the same time?

Let x stand for the number of minutes after which the cistern will be filled. In one minute $\frac{1}{24}$ of its capacity enters through the first pipe, and hence in x minutes $\frac{1}{24}x$ of its capacity enters. For a similar reason, $\frac{1}{36}x$ of its capacity enters through the second pipe in x minutes; and in the same time $\frac{1}{a}x$ of its capacity is discharged through the third pipe.

Therefore, after x minutes there is in the cistern

$$\frac{x}{24} + \frac{x}{36} - \frac{x}{a}, = (\tfrac{5}{72} - a)x,$$

of its capacity. But by the condition of the problem, that the cistern is then filled, we have

$$(\tfrac{5}{72} - a)x = 1;$$

whence

$$x = \frac{1}{\frac{5}{72} - a}.$$

If we now let a approach $\frac{5}{72}$, then x becomes infinite.

This result would mean that the cistern would never be filled. This is also evident from the data of the problem, since the third pipe in a given time would discharge from the cistern as much as would enter it through the other pipes.

The Problem of the Couriers.

24. Pr. Two couriers are travelling along a road in the direction from M to N; one courier at the rate of m_1 miles an hour, the other at the rate of m_2 miles an hour. The former

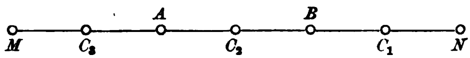


FIG. 2.

is seen at the station A at noon, and the other is seen h hours later at the station B, which is d miles from A in the direction in which the couriers are travelling. Where do the couriers meet?

Assume that they meet to the right of B at a point C_1, and let x stand for the number of miles from B to the place of meeting C_1 (Fig. 2).

The first courier, moving at the rate of m_1 miles an hour, travels $d+x$ miles, from A to C_1, in $\frac{d+x}{m_1}$ hours; the second courier, moving at the rate of m_2 miles an hour, travels x miles, from B to C_1, in $\frac{x}{m_2}$ hours. By the condition of the problem it is evident that, if the place of meeting is to the right of B, the number of hours it takes the first courier to travel from A to C_1 exceeds by h the number of hours it takes the second courier to travel from B to C_1. We therefore have

$$\frac{d+x}{m_1} - \frac{x}{m_2} = h,$$

whence $$x = \frac{hm_1m_2 - dm_2}{m_2 - m_1} = \frac{m_2(hm_1 - d)}{m_2 - m_1}.$$

(i.) **A Positive Result.** — The result will be positive either when $hm_1 > d$ and $m_2 > m_1$, or when $hm_1 < d$ and $m_2 < m_1$. A positive result means that the problem is possible with the assumption made; *i.e.*, that the couriers meet at a point to the right of B.

(ii.) **A Negative Result.** — The result will be negative either when $hm_1 > d$ and $m_2 < m_1$, or when $hm_1 < d$ and $m_2 > m_1$. Such

a result shows that the assumption that the couriers meet to the right of B is untenable, since, as we have seen, in that case the result is positive.

That under the assumed conditions the couriers can meet only at some point to the left of B can also be inferred from the following considerations, which are independent of the negative result: If $hm_1 > d$, the first courier has passed B when the second courier is seen at that station; that is, the second courier is behind the first at that time. And since also $m_2 < m_1$, the first courier is travelling the faster, and must therefore have overtaken the second, and at some point to the left of B.

On the other hand, if $hm_1 > d$, the first courier has not yet reached B when the second is seen at that station; that is, the first courier is behind the second at that time. And since also $m_2 > m_1$, the second courier is travelling the faster, and must therefore have overtaken the first, at some point to the left of B. Similar reasoning could have been applied in (i.).

(iii.) **A Zero Result.** — A zero result is obtained when $hm_1 = d$, and m_2 is not equal to m_1; that is, the meeting takes place at B. This is also evident from the assumed conditions. For the first courier reaches B h hours after he was seen at A; and since the second courier is seen at B h hours after the first was seen at A, the meeting must take place at B.

(iv.) **Indeterminate Result.** — An indeterminate result is obtained if $hm_1 \doteq d$, and $m_2 \doteq m_1$. In this case every point of the road can be regarded as their place of meeting. For the first courier evidently reaches B at the time at which the second courier is seen at that station; and since they are travelling at the same rate, they must be together all the time. The problem under these conditions becomes indeterminate.

(v.) **An Infinite Result.** — An infinite result is obtained when $hm_1 \neq d$, and $m_2 \doteq m_1$. In this case a meeting of the couriers is impossible, since both travel at the same rate, and when the second is seen at B the first either has not yet reached B or has already passed that station.

An infinite result also means that the more nearly equal m_1 and m_2 are, the further removed is the place of meeting.

EXERCISES III.

Solve the following problems, and interpret the results:

1. In a number of two digits, the digit in the tens' place exceeds the digit in the units' place by 5. If the digits be interchanged, the resulting number will be less than the original number by 45. What is the number?

2. The sum of the first and third of three consecutive even numbers is equal to twice the second. What are the numbers?

3. A father is 26 years older than his son, and the sum of their ages is 26 years less than twice the father's age. How old is the son?

4. In a number of two digits, the digit in the units' place exceeds the digit in the tens' place by 4. If the sum of the digits be divided by 2, the quotient will be less than the first digit by 2. What is the number?

Discuss the solutions of the following general problems:

5. What number, added to the denominators of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will make the resulting fractions equal?

6. Having two kinds of wine worth a and b dollars a gallon, respectively, how many gallons of each kind must be taken to make a mixture of n gallons worth c dollars a gallon?

7. Two couriers, A and B, start at the same time from two stations, distant d miles from each other, and travel in the same direction. A travels n times as fast as B. Where will A overtake B?

CHAPTER XXV.

INFINITE SERIES.

1. The infinite series

$$1 + \tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots$$

is a decreasing geometrical progression, whose ratio is $\tfrac{2}{3}$.

Let $\quad S_n = 1 + \tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots$ to n terms.

Then, by Ch. XXI., Art. 26,

$$S_n \doteq \frac{a}{1-r}, = \frac{1}{1-\tfrac{2}{3}}, = 3,$$

as n increases indefinitely. By actual computation, we obtain

$$S_1 = 1,\ S_2 = 1\tfrac{2}{3},\ S_3 = 2\tfrac{1}{9},\ S_4 = 2\tfrac{11}{27}, \text{ etc.}$$

These sums approach 3 more and more nearly, as more and more terms are included. This infinite series may therefore be regarded as having the finite sum 3.

But the sum of the series

$$1 + 2 + 4 + 8 + \cdots$$

increases beyond any finite number, however great, as the number of terms increases indefinitely.

2. The examples of the preceding article illustrate the following definitions:

An infinite series is said to be **Convergent**, when the sum of the first n terms approaches a definite finite limit, as n increases indefinitely.

An infinite series is said to be **Divergent**, when the sum of the first n terms increases numerically beyond any assigned number, however great, as n increases indefinitely.

3. It was shown in Ch. XXI., Art. 26, that the sum of n terms of the geometrical progression

$$a + ar + ar^2 + \cdots,$$

when $r < 1$, approaches the definite finite limit $\frac{a}{1-r}$, as n increases indefinitely.

Therefore, *any decreasing geometrical progression is a convergent series.*

4. Infinite series arise in connection with many mathematical operations. Thus, for example, if the division of 1 by $1 - x$ be continued indefinitely, we obtain as a quotient the infinite series

$$1 + x + x^2 + x^3 + \cdots.$$

When x is numerically less than 1, that is, lies between -1 and 1, this series is a decreasing geometrical progression, as in Art. 1. Therefore, by the preceding article, it is convergent.

Thus, when $x = \frac{2}{3}$, as in Art. 1, the sum of n terms of the series approaches 3, as n increases indefinitely; and

$$\frac{1}{1-x} = \frac{1}{1-\frac{2}{3}} = 3.$$

Consequently, we may take the series as the expansion of the fraction, for all values of x between -1 and 1, and write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots,$$

for such values of x.

When $x = 1$, the series becomes

$$1 + 1 + 1 + \cdots,$$

and is evidently divergent. If we assign as the value of the fraction, when $x = 1$, the limit of the fraction as x approaches 1, as in Ch. XXIV., Art. 19, we have

$$\frac{1}{1-x} = \infty,$$

when $x = 1$. Since both the fraction and the sum of the series are infinite when $x = 1$, in this sense we may assume that they are equivalent.

When $x = -1$, we have

$$1 - 1 + 1 - 1 + \cdots.$$

The sum of n terms of this series is 1 or 0, according as n is odd or even. The series is said to *oscillate*, and is neither convergent nor divergent. But, when $x = -1$,

$$\frac{1}{1-x} = \frac{1}{1+1} = \frac{1}{2}.$$

Consequently, we cannot assume that the series is the expansion of the fraction when $x = -1$.

When x is greater than 1, numerically, we have

$$S_n = \frac{1-x^n}{1-x}.$$

By taking n sufficiently great, x^n, and therefore $\frac{1-x^n}{1-x}$ can be made to exceed numerically any number, however great. Therefore, the series is divergent.

Thus, when $x = 2$, the series becomes

$$1 + 2 + 4 + 8 + \cdots.$$

But, when $x = 2$, $\quad \frac{1}{1-x} = \frac{1}{1-2} = -1.$

Therefore, we cannot assume that the series is the expansion of the fraction when x is numerically greater than 1.

In general, an infinite series, no matter how obtained from a given expression, can be regarded as the expansion of the expression, for values of x which make the latter finite, *only when the series is convergent for such values of* x.

5. In the preceding article the convergency or divergency of the series was determined by an examination of the formula for the sum of n terms. There are, however, many infinite series for which such formulæ have not been obtained. In such cases it is necessary to determine the convergency or divergency of the series by other methods. Even when a formula for the sum of n terms is known, methods now to be given are often to be preferred.

6. In the theory which follows, we shall let S stand for the limit of the sum of n terms of the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

as n increases indefinitely.

Also let $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, the sum of n terms,

and $${}_mR_n = u_{n+1} + u_{n+2} + \cdots + u_{n+m},$$

the sum of m terms after the first n terms.

Then, evidently, $S_n + {}_mR_n = S_{n+m}$, and $\lim_{n \doteq \infty} S_n = S$.

7. *The series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent if S_n *remain finite, and* ${}_mR_n$ *approach* 0 *for all values of* m, *as* n *increases indefinitely; and, conversely, if the series be convergent, these two conditions are satisfied.*

By the first condition S_n is finite. By the second condition,

$$\lim\,(S_{n+m} - S_n), = \lim\,{}_mR_n, = 0,$$

as n increases indefinitely.

Therefore, $$\lim S_{n+m} = \lim S_n.$$

That is, S_n cannot have one finite limit for one value of n, and a different finite limit for another value of n. Hence the limit of S_n is a *definite* finite number, and the series is convergent.

If, conversely, the series be convergent, the limit of S_n must be a *definite* finite number, and

$$\lim S_{n+m} = \lim S_n.$$

Hence $$\lim\,(S_{n+m} - S_n), = \lim\,{}_mR_n, = 0.$$

This principle is to be applied when it is possible to prove that the limit of the sum of n terms is finite, but not that it is a *definite* finite number. If, in addition, it can be proved that $\lim {}_mR_n = 0$, this deficiency is supplied.

The oscillating series

$$1 - 1 + 1 - 1 + \cdots$$

is an instance of series which satisfy the first condition of the principle, but not the second. The limit of the sum of n terms is, as we have seen, 1 or 0, and is therefore finite.

Let $n = 2k$, an even number.

Then, $\lim (S_{2k+1} - S_{2k}), = {}_1R_{2k}, = 1$, not 0.

8. The following principle also does away with the necessity of proving that the limit of the sum of n terms is *definite* as well as finite, when the terms of the series are all positive.

If the sum of n terms of an infinite series of positive terms remain finite, as n increases indefinitely, the series is convergent.

For, since the sum continually increases, but remains finite, it must ultimately differ from some definite finite number by less than any assigned number, however small. This definite finite number is therefore the limit of the sum.

9. *If a finite number of terms be added to, or subtracted from, a given convergent series, the resulting series will be convergent; if added to, or subtracted from, a given divergent series, the resulting series will be divergent.*

For, the sum of a finite number of terms is a definite finite number. If this sum be added to the finite limit of the sum of n terms of a convergent series, the resulting sum will be a definite finite number, and the series therefore convergent.

In a similar manner the second part of the principle can be proved.

Methods of Comparison.

10. In the principles which now follow, it will be assumed that the terms of the series are all positive, unless the contrary is stated.

11. *If, after some particular term of an infinite series, each term be less than the corresponding term of a series known to be convergent, the given series is convergent.*

For, beginning with some particular term, which may or may not be the first term, the sum of n terms of the given series is less than the sum of the corresponding n terms of the known convergent series. This sum is therefore finite. Hence, by Art. 8, the series, beginning with some particular term, is convergent. It then follows from Art. 9, that the given series is convergent.

Ex. Compare the series

$$1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots, \quad =1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots,$$

with the known convergent series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots, \quad =1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\cdots.$$

Evidently, *after the second term*, the denominator of each term of the given series is *greater* than the denominator of the corresponding term of the second series, and therefore each term of the given series, after the second, is *less* than the corresponding term of the second series.

Hence the given series is convergent.

12. *If, after some particular term of a given infinite series, each term be greater than the corresponding term of a known divergent series, the given series is divergent.*

The proof of this principle is similar to that of the preceding article.

Ex. Compare the series $\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots$ with the known divergent series $1+1+1+\cdots$.

Each term of the given series is greater than the corresponding term of the second series. Hence the given series is divergent.

13. In applying the principles of Arts. 11–12, certain series are important. These we now discuss.

(i.) Examine the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots.$$

We have
$$1+\frac{1}{2}=1+\frac{1}{2};$$
$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}, =\frac{1}{2};$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}, =\frac{1}{2};$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Whence, by addition,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

The series in the second member is evidently divergent, and hence, with greater reason, the given series is divergent.

(ii.) The preceding series is a particular instance of the series

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots$$

wherein k is assumed to be positive.

We will first show for what values of k this series is convergent, by comparing it with a series of greater terms.

$$1=1;$$
$$\frac{1}{2^k}+\frac{1}{3^k}<\frac{1}{2^k}+\frac{1}{2^k}, =\frac{2}{2^k}, =\frac{1}{2^{k-1}};$$
$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}<\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}, =\frac{1}{4^{k-1}};$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Whence, by addition,

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots<1+\frac{1}{2^{k-1}}+\frac{1}{4^{k-1}}+\frac{1}{8^{k-1}}\cdots.$$

The series in the second member is a geometrical progression whose ratio is $(\frac{1}{2})^{k-1}$. When $k>1$, $k-1$ is positive, integral or fractional. Therefore $(\frac{1}{2})^{k-1}$ is less than 1, and the series in the second member is convergent, by Art. 3. Consequently, by Art. 8, the given series is convergent, when $k>1$.

E.g., the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ is convergent.

When $k=1$, the series is that which was proved to be divergent in (i.).

When $k<1$, $2^k<2$, and therefore $\frac{1}{2^k}>\frac{1}{2}$.

In like manner, $\frac{1}{3^k}>\frac{1}{3}$, $\frac{1}{4^k}>\frac{1}{4}$, $\cdots$.

Therefore,

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots.$$

Since the series in the second member is divergent, the given series is divergent, when $k<1$.

E.g., $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$, $=1+\frac{1}{2^{\frac{1}{2}}}+\frac{1}{3^{\frac{1}{2}}}+\frac{1}{4^{\frac{1}{2}}}+\cdots$,
is a divergent series.

We therefore conclude:

The series $1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots$

is convergent when $k>1$; and divergent when $k=1$, or $k<1$.

14. In thus comparing one series with another it is important not to be misled by the relative values of the first few terms of the two series.

Thus, compare the terms of the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$$

with the corresponding terms of the series

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots.$$

We have $\quad \frac{1}{2\cdot 3}<\frac{1}{2^2}, \ \frac{1}{3\cdot 4}<\frac{1}{2^3}, \ \frac{1}{4\cdot 5}<\frac{1}{2^4},$

but $\frac{1}{5\cdot 6}>\frac{1}{2^5}$; and so for all subsequent terms. Had we given attention only to the first four terms, we should have inferred that the first series is convergent from a comparison with the second series, whereas the question of its convergency or divergency is not settled. Now compare the given series with the known convergent series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots, \quad =1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\cdots.$$

It is evident, from the forms of the denominators in the two series, that each term of the given series, after the first, is less than the corresponding term of the last series. Hence the given series is convergent.

EXERCISES I.

Determine the convergency or divergency of the series:

1. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\cdots.$

2. $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{3}{4}}+\cdots.$

3. $\frac{3}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots.$

4. $\frac{2}{1\cdot 3}+\frac{3}{2^2\cdot 5}+\frac{4}{3^2\cdot 7}+\cdots.$

5. $\frac{2}{3}+\frac{2\cdot 3}{3\cdot 5}+\frac{2\cdot 3\cdot 4}{3\cdot 5\cdot 7}+\cdots.$

6. $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\frac{4}{3\cdot 5}+\cdots.$

7. $\frac{2}{1}+\frac{3}{2^2}+\frac{4}{3^2}+\cdots.$

8. $1+\frac{3}{\lfloor 2}+\frac{3^2}{\lfloor 3}+\cdots.$

9. $\frac{1}{1+2}+\frac{1}{1+2^2}+\frac{1}{1+2^3}+\cdots.$

10. $\frac{a+x}{b+x}+\frac{(a+x)(2a+x)}{(b+x)(2b+x)}+\frac{(a+x)(2a+x)(3a+x)}{(b+x)(2b+x)(3b+x)}+\cdots,$

wherein a, b, and x are positive, and $b>a$.

The General Term of a Series.

15. If the general term, the nth say, be given, the entire series is known.

Ex. Write the series whose nth term is

$$\frac{2^{n-1}}{(2n-1)(2n+1)} \cdot \frac{x^{4n+1}}{4n+1}.$$

Giving to n the series of values 1, 2, 3, $\cdots$, we obtain

$$\frac{1}{1 \cdot 3} \cdot \frac{x^5}{5} + \frac{2}{3 \cdot 5} \cdot \frac{x^9}{9} + \frac{2^2}{5 \cdot 7} \cdot \frac{x^{13}}{13} + \cdots.$$

16. It is frequently necessary to write the nth term of a series, when only the first few terms are given.

Ex. Write the nth term of the series

$$1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots.$$

The exponent of x in the second term is $2 \times 2 - 2, = 2$; in the third term, $2 \times 3 - 2, = 4$; and in the nth, $2n - 2$.

The first factor in the numerator of each term after the first is 1, and the last is one less than the exponent of x. Hence the numerator of the nth term is $1 \cdot 3 \cdot 5 \cdots (2n-3)$.

In the denominators, after the first term, the first factor is 2, and the last is the same as the exponent of x. Hence the denominator of the nth term is $2 \cdot 4 \cdot 6 \cdots (2n-2)$.

Therefore, the nth term is $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} x^{2n-2}$.

Observe that only the terms after the first are obtained from the nth term as thus written.

17. *If the ratio of each term of a given infinite series to the corresponding term of another infinite series be finite, the given series is convergent when the second series is convergent, and is divergent when the second series is divergent.*

Let $$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$
be the given series,

and $$v_1 + v_2 + v_3 + \cdots + v_n + \cdots$$
be a series known to be convergent, or divergent.

First, let the second series be convergent.

Let k be a finite number greater than the greatest of the finite ratios
$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \cdots, \frac{u_n}{v_n}, \cdots.$$

Then $$\frac{u_1}{v_1} < k, \frac{u_2}{v_2} < k, \frac{u_3}{v_3} < k, \cdots, \frac{u_n}{v_n} < k, \cdots;$$

whence, $u_1 < kv_1$, $u_2 < kv_2$, $u_3 < kv_3$, $\cdots$, $u_n < kv_n$, $\cdots$.

By addition, we have
$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots < k\,(v_1 + v_2 + v_3 + \cdots + v_n + \cdots).$$

Since $v_1 + v_2 + \cdots + v_n$ remains finite as n increases indefinitely, and k is finite, $u_1 + u_2 + u_3 + \cdots + u_n$ remains finite, as n increases indefinitely. Hence, by Art. 8, the given series is convergent.

Next, let the second series be divergent.

Now let k be a finite number less than the least of the finite ratios
$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \cdots, \frac{u_n}{v_n}, \cdots.$$

Then, we readily obtain
$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots > k(v_1 + v_2 + v_3 + \cdots + v_n + \cdots).$$

Since the second series is divergent, $v_1 + v_2 + v_3 + \cdots + v_n$, and with greater reason, $u_1 + u_2 + u_3 + \cdots + u_n$, increases beyond any assigned number, however great, as n increases indefinitely.

Therefore the given series is divergent.

Ex. 1. Examine the series
$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4\cdot} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots$$

Compare this series with the known convergent series [Art. 13 (ii.)]
$$\frac{1}{1} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \cdots.$$

It is sufficient to examine the ratio of the nth term of the given series to the nth term of the second series.

This ratio is

$$\frac{1}{n\,(n+1)(n+2)} \div \frac{1}{n^3}, = \frac{n^3}{(n+1)(n+2)}, \doteq 1,$$

as n increases indefinitely. Since, therefore, the ratio is always finite, the given series is convergent.

Ex. 2. Examine the series

$$\frac{2}{1 \cdot 3} + \frac{4}{3 \cdot 5} + \frac{6}{5 \cdot 7} + \cdots + \frac{2\,n}{(2\,n-1)(2\,n+1)} + \cdots.$$

Compare with the known divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

The ratio of the nth terms is

$$\frac{2\,n}{(2\,n-1)(2\,n+1)} \div \frac{1}{n}, = \frac{2\,n^2}{(2\,n-1)(2\,n+1)}, \doteq \frac{1}{2},$$

as n increases indefinitely.

Since this ratio is finite for all values of n, the given series is divergent.

EXERCISES II.

Determine the convergency or divergency of the series whose nth terms are:

1. $\dfrac{2\,n-5}{n^3-5\,n}$.
2. $\dfrac{1+n}{1+n^2}$.
3. $\dfrac{n+2}{n^3+1}$.
4. $\dfrac{n^2-(n-1)^2}{n^2+(n+1)^2}$.
5. $\dfrac{(n+a)(n+b)}{n(n+1)(n+2)}$.

Determine the convergency or divergency of the series:

6. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots$.
7. $\dfrac{1}{2} + \dfrac{2}{3\sqrt{2}} + \dfrac{3}{4\sqrt{3}} + \cdots$.
8. $\dfrac{3}{1 \cdot 2} + \dfrac{5}{2^2 \cdot 3} + \dfrac{7}{3^2 \cdot 4} + \cdots$.
9. $\dfrac{9}{2 \cdot 3} + \dfrac{16}{3 \cdot 4} + \dfrac{25}{4 \cdot 5} + \cdots$.
10. $\dfrac{1}{a(a+b)} + \dfrac{1}{(a+2\,b)(a+3\,b)} + \dfrac{1}{(a+4\,b)(a+5\,b)} + \cdots$.

Series having Negative Terms.

18. *If a series be convergent when its terms are all positive, it will remain convergent when some or all of its terms are made negative.*

Since S_n remains finite and ${}_mR_n \doteq 0$, when all the terms are positive, with greater reason S_n will remain finite and ${}_mR_n \doteq 0$, when some or all of the terms are made negative.

Ex. The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ is convergent,
since the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is convergent.

19. A series which is convergent when all its negative terms are made positive is said to be **Absolutely Convergent.**

Evidently every convergent series whose terms are all positive is absolutely convergent.

20. *If the terms of an infinite series be alternately positive and negative, and the nth term approach* 0, *as n increases indefinitely, the series is convergent.*

Let the given series be

$$u_1 - u_2 + u_3 - \cdots + (-1)^{n-1}u_n + \cdots.$$

Then
$$S_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n$$
$$= (u_1 - u_2) + (u_3 - u_4) + \cdots \tag{1}$$
$$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots. \tag{2}$$

Since the terms decrease numerically, it is evident that in (1) and (2) the sums inclosed in the parentheses are positive. Therefore from (1) we infer that S_n is *positive,* and from (2) that it is *less than the first term* u_1. Therefore S_n is finite.

Also,

$${}_mR_n = (-1)^n[u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \cdots + (-1)^{m-1}u_{n+m}]$$
$$= (-1)^n[(u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \cdots] \tag{3}$$
$$= (-1)^n[u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots]. \tag{4}$$

From (3) we infer that the part of ${}_mR_n$ in the brackets is positive, and from (4) that it is less than u_{n+1}. Since $u_{n+1} \doteq 0$, it follows that ${}_mR_n \doteq 0$. Hence the given series is convergent.

Ex. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, but not absolutely convergent, since $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

The Ratio of Convergency.

21. In the following principle the terms of the series are not necessarily all positive:

An infinite series is convergent, if, after some particular term, the ratio of each term to the preceding be numerically less than some fixed positive number, which is itself less than unity.

Let the given series be

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

and let the ratio of each term after the kth to the preceding be less than r, which is itself less than 1.

First, assume that the terms are all positive.

Then, from $\dfrac{u_{k+1}}{u_k} < r,\ \dfrac{u_{k+2}}{u_{k+1}} < r,\ \dfrac{u_{k+3}}{u_{k+2}} < r, \cdots;$

we obtain

$$u_{k+1} < ru_k,\ u_{k+2} < ru_{k+1} < r^2u_k,\ u_{k+3} < ru_{k+2} < r^3u_k, \cdots.$$

Whence, by addition,

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots < u_k(r + r^2 + r^3 + \cdots).$$

Since $r < 1$, $r + r^2 + r^3 + \cdots \doteq \dfrac{r}{1-r}$, a finite number.

Therefore, since the sum of the finite number of terms $u_1 + u_2 + \cdots + u_{k-1}$ is finite, the given series is, by Art. 8, convergent.

When some or all of the terms are negative, the series is, by Art. 18, convergent.

22. *An infinite series of positive terms is divergent, if, after some particular term, the ratio of each term to the preceding be equal to unity, or greater than unity.*

In the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

let the ratio of each term after the kth be equal to 1. Then the sum of n terms, after the kth, is equal to nu_k, and hence increases beyond any assigned number however great, as n increases indefinitely.

Next, let this ratio be greater than unity; then the sum of n terms after the kth is greater than nu_k, and hence increases beyond any assigned number, however great, as n increases indefinitely.

Therefore, in each case, the series is divergent.

23. The ratio of the nth term to the preceding is called the **Ratio of Convergency** of the series.

24. The following examples will illustrate the principles of Arts. 21–22.

Ex. 1. Examine the series

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \cdots.$$

The ratio of convergency is

$$\frac{n}{2^n} \div \frac{n-1}{2^{n-1}}, \; = \frac{n}{n-1} \cdot \frac{1}{2}, \; = \frac{1}{1 - \frac{1}{n}} \cdot \frac{1}{2}, \; \doteq \frac{1}{2}.$$

By taking n large enough we can make this ratio differ from $\frac{1}{2}$ by as little as we please, and consequently less than some number between $\frac{1}{2}$ and 1; that is, less than some number which is itself less than 1.

Thus, if $n = 4$, the ratio is equal to $\frac{2}{3}$, which is less than, say $\frac{3}{4}$. That the ratio will remain less than $\frac{3}{4}$ for values of n greater than 4, can be shown as follows.

Assume $\dfrac{n}{n-1}\cdot\dfrac{1}{2}<\dfrac{3}{4}$; then $2n<3n-3$, and $n>3$.

Since, therefore, after the third term, the ratio of each term to the preceding is less than $\frac{3}{4}$, which is less than 1, the given series is convergent.

Ex. 2. Examine the series

$$\frac{1\cdot 3}{2}+\frac{3\cdot 5}{2^2}x+\frac{5\cdot 7}{2^3}x^2+\cdots\frac{(2n-1)(2n+1)}{2^n}x^{n-1}+\cdots.$$

The ratio of convergency is

$$\frac{(2n-1)(2n+1)}{2^n}x^{n-1}\div\frac{(2n-3)(2n-1)}{2^{n-1}}x^{n-2}$$
$$=\frac{(2n+1)x}{(2n-3)2},\ \doteq\frac{x}{2}.$$

By taking n sufficiently great, we can make this ratio differ from $\frac{1}{2}x$ by as little as we please. If, therefore, x have a definite value less than 2, the ratio can be made less than some number, say k, which is itself less than 1. Hence the series is convergent when $x<2$.

The term after which this ratio becomes and remains less than k is determined from

$$\frac{2n+1}{2n-3}\cdot\frac{x}{2}<k,\ \text{whence}\ n>\frac{6k+x}{2(2k-x)}.$$

Thus, let $x=\frac{3}{2}$, or $\frac{1}{2}x=\frac{3}{4}$, and $k=\frac{5}{6}$. We find $n>19\frac{1}{2}$. That is, when $x=\frac{3}{2}$, the ratio of each term, after the 19th, to the preceding is less than $\frac{5}{6}$, which is less than 1.

Evidently, when $x=2$, or $x>2$, the ratio is greater than 1 for all values of n. Therefore the series is then divergent.

25. The significance of the words, *less than some number which is itself less than unity*, is shown by an examination of the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots,$$

which is known to be divergent.

The ratio of convergency is $\frac{n-1}{n}, = 1 - \frac{1}{n}.$

This ratio is less than 1, but by taking n large enough, it can be made to differ from 1 by as little as we please.

The value of this ratio will therefore not remain less than some definite number, which is itself less than 1. This condition of the principle of Art. 21 is not satisfied, and the test fails. Also, since neither condition of Art. 22 is satisfied, the test fails to prove the series divergent. In such cases, it is necessary to try other tests, just as the above series was by other means proved to be divergent.

26. It is not, in general, necessary to determine the number of the term after which the ratio of any term to the preceding is less than some definite number which is itself less than 1, in the case of a convergent series. The following method, illustrated by the examples of the preceding articles, is sufficient:

Determine the limit of the ratio of convergency as n increases indefinitely.

(i.) *If this limit* < 1, *the series is convergent.*

(ii.) *If this limit* > 1, *the series is divergent.*

(iii.) *If this limit* $= 1$, *the convergency or divergency of the series is, as a rule, not settled, and some other test must be applied.*

But, if the ratio be always greater than 1, *as it approaches the limit* 1, *the series is, by Art.* 22, *divergent.*

Ex. Examine the series

$$\frac{4 \cdot 5x}{1 \cdot 2 \cdot 3} + \frac{5 \cdot 6x^2}{2 \cdot 3 \cdot 4} + \cdots + \frac{(n+3)(n+4)x^n}{n(n+1)(n+2)} + \cdots.$$

The ratio of convergency is

$$\frac{(n+3)(n+4)x^n}{n(n+1)(n+2)} \div \frac{(n+2)(n+3)x^{n-1}}{(n-1)n(n+1)},$$

$$= \frac{(n+4)(n-1)}{(n+2)(n+2)} \cdot x, \doteq x,$$

as n increases indefinitely.

Hence, for values of $x<1$, the series is convergent; for values of $x>1$, the series is divergent; while, for $x=1$, the series is in doubt. When $x=1$, we have

$$\frac{4\cdot 5}{1\cdot 2\cdot 3}+\frac{5\cdot 6}{2\cdot 3\cdot 4}+\cdots+\frac{(n+3)(n+4)}{n(n+1)(n+2)}+\cdots.$$

We will try the method of Art. 17, comparing with the known divergent series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots.$$

The ratio of the nth term of the given series to the nth term of the auxiliary series is

$$\frac{(n+3)(n+4)}{n(n+1)(n+2)}\div\frac{1}{n},=\frac{(n+3)(n+4)}{(n+1)(n+2)},\doteq 1.$$

This ratio is evidently finite for all values of n. Therefore, when $x=1$, the given series is divergent.

27. The following application of the principle of Art. 21 will be required in Ch. XXVII.

The series

$$1+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots$$

is absolutely convergent, when <1 *numerically.*

In the above series n is finite. We will therefore take the ratio of the $(k+1)$th term to the preceding.

The ratio of convergence is

$$\frac{n(n-1)\cdots(n-k+1)}{\underline{|k}}x^k\div\frac{n(n-1)\cdots(n-k+2)}{\underline{|k-1}}x^{k-1}$$

$$=\frac{n-k+1}{k}x,\doteq -x,$$

as k increases indefinitely.

Therefore, the series is absolutely convergent, when $x<1$ numerically.

EXERCISES III.

Determine the convergency or divergency of the series:

1. $1 + \frac{2^k}{|\underline{2}} + \frac{3^k}{|\underline{3}} + \cdots.$

2. $\frac{2}{1} + \frac{2 \cdot 3}{1 \cdot 3} + \frac{2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5} + \cdots.$

3. $\frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 7} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 7 \cdot 10} + \cdots.$

4. $\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 6} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 6 \cdot 9} + \cdots.$

5. $\frac{1}{a+1} + \frac{k}{a+k} + \frac{k^2}{a+2k} + \cdots.$

Determine for what values of x the following series are convergent or divergent:

6. $1^2 + 2^2x + 3^2x^2 + \cdots.$

7. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots.$

8. $1 + \frac{x}{|\underline{1}} + \frac{x^2}{|\underline{2}} + \cdots.$

9. $\frac{1}{1 \cdot 2} + \frac{x}{2 \cdot 3} + \frac{x^2}{3 \cdot 4} + \cdots.$

10. $\frac{\pi}{1} - \frac{1}{x} + \frac{1}{3x^3} - \cdots.$

11. $\frac{1}{1 \cdot 3} + \frac{2x}{3 \cdot 5} + \frac{(2x)^2}{5 \cdot 7} + \cdots.$

12. $1 - \frac{3x}{2^2} + \frac{5x^2}{3^2} - \cdots.$

13. $1 + \frac{4x}{5} + \frac{9x^2}{5^2} + \cdots.$

14. $1 + \frac{3^3x}{|\underline{2}} + \frac{5^3x^2}{|\underline{3}} + \cdots.$

15. $1 + 2^2x + \frac{3^2x^2}{|\underline{2}} + \cdots.$

16. $a + (a+d)x + (a+2d)x^2 + \cdots.$

17. $\frac{1}{1+x} + \frac{1}{1+x^2} + \frac{1}{1+x^3} + \cdots.$

18. $\frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \cdots.$

19. $\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \cdots.$

20. $1 - \frac{x}{1+k} + \frac{x^2}{1+2k} - \cdots.$

CHAPTER XXVI.

UNDETERMINED COEFFICIENTS.

1. Upon the following principles is based an important method of changing an algebraical expression from one form to another.

2. *If an infinite series* $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ *be convergent for values of* x *greater than* 0, *the sum of the series approaches* a_0, *as* x *approaches* 0.

Let $$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = a_0 + xS_1,$$

wherein $$S_1 = a_1 + a_2x + a_3x^2 + \cdots.$$

Evidently, if the given series be convergent, that is, if $a_0 + xS_1$ be finite, then S_1 is finite. Therefore, by Ch. XXIV., Art. 16, $xS_1 \doteq 0$, when $x \doteq 0$.

Consequently

$$a_0 + a_1x + a_2x^2 + \cdots, = a_0 + xS, \doteq a_0, \text{ when } x \doteq 0.$$

3. *If two integral series, arranged to ascending powers of* x, *be equal for all values of* x *which make them both convergent, the coefficients of like powers of* x *are equal.*

Let $$a_0 + a_1x + a_2x^2 + \cdots = b_0 + b_1x + b_2x^2 + \cdots$$

for all values of x which make the two series convergent.

Then the sums of the two series approach equal limits when $x \doteq 0$. But, by the preceding article, the sum of the one series approaches a_0, that of the other b_0; consequently $a_0 = b_0$,

and $$a_1x + a_2x^2 + \cdots = b_1x + b_2x^2 + \cdots.$$

Since by Ch. XXV, Art. 21, these two series are convergent for all values of x for which the original series are convergent, they are equal for values of x other than zero, and the last equation may be divided by x.

Hence $a_1 + a_2x + a_3x^2 + \cdots = b_1 + b_2x + b_3x^2 + \cdots;$

and as before, $a_1 = b_1,$

and $a_2x + a_3x^2 + \cdots = b_2x + b_3x^2 + \cdots.$

In like manner, we can prove $a_2 = b_2$, $a_3 = b_3$, etc.

4. The principle of Art. 3 holds with greater reason if either or both of the series be finite. The series must be equal for all values of x, if they be both finite; or, if one be infinite, for all values of x which make that series convergent.

5. The condition that the roots of the equation

$$ax^2 + bx + c = 0$$

are equal, given in Ch. XVIII., Art. 12 (ii.), can be obtained also by applying the principle of Art. 3.

If the two roots be equal, $ax^2 + bx + c$ is the square of a binomial. We therefore assume

$$ax^2 + bx + c = (Ax + B)^2 = A^2x^2 + 2\,ABx + B^2.$$

By Art. 3, $A^2 = a$ (1), $2\,AB = b$ (2), $B^2 = c$ (3).

From (1) and (3), $A = \sqrt{a}$, $B = \sqrt{c}$.

Whence, by (2), $2\sqrt{(ac)} = b$, or $b^2 = 4\,ac$.

Expansion of Rational Fractions.

6. We shall now give a method of expanding a fraction in an infinite series, without performing the actual division.

Ex. 1. Expand $$\frac{2-x}{1+x-x^2},$$

in a series, to ascending powers of x.

We equate the fraction to a series of the required form, in which the coefficients of the different powers of x are unknown, or *undetermined*.

Assume $$\frac{2-x}{1+x-x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots,$$

wherein $A, B, C, D, E, \cdots$ are constants to be determined.

Clearing the equation of fractions, we obtain

$$\begin{array}{rl|l|l|l|l} 2-x= & A+B & x+C & x^2+D & x^3+E & x^4+\cdots. \\ & A & +B & +C & +D & \\ & & -A & -B & -C & \end{array}$$

In this work the powers of x in the terms of the second and third partial products are omitted, it being understood that the letters remaining are the coefficients of the powers of x just above in the first partial product.

Thus the coefficient of x is $A+B$, etc.

The series on the right is infinite; that on the left may be regarded as an infinite series with zero coefficients of all powers of x higher than the first. By Art. 3, we have

$$\begin{array}{lll} A=2; & B+A=-1, & \text{whence } B=-3; \\ & C+B-A=0, & \text{whence } C=5; \\ & D+C-B=0, & \text{whence } D=-8; \\ & E+D-C=0, & \text{whence } E=13; \\ & \text{etc.,} & \text{etc.} \end{array}$$

Hence, substituting these values of $A, B, C, D, \cdots$ in the assumed series, we have

$$\frac{2-x}{1+x-x^2}=2-3x+5x^2-8x^3+13x^4+\cdots.$$

We can assume that this series is equivalent to the fraction only when x has such values as make it convergent.

Let the student compare this result with that obtained by division. In fact, the latter method of expanding a fraction is to be preferred when only a few terms are wanted. But the successive coefficients, after a certain stage, may be computed with great facility by the *method of undetermined coefficients.* A moment's inspection of the preceding work will convince the student that the coefficient D, and all which follow it, are each connected with the two immediately preceding coefficients by a definite relation. Thus,

$$D+C-B=0,\ E+D-C=0,\ F+E-D=0, \text{ etc.}$$

In assuming as the expansion of a rational fraction an infinite series of ascending powers of x, it is usually necessary first to determine with what power the series should commence. This is done by division, when both numerator and denominator are arranged to ascending powers of x. In fact, this step also determines completely the first term of the series.

Ex. 2. Expand $$\frac{1-x}{3x^2-x^3},$$
in a series to ascending powers of x.

The first term in the expansion, obtained by division, is evidently $\frac{1}{3}x^{-2}$.

We therefore assume

$$\frac{1-x}{3x^2-x^3}=\tfrac{1}{3}x^{-2}+Bx^{-1}+C+Dx+Ex^2+Fx^3+\cdots.$$

Clearing of fractions, we obtain

$$1-x=1+\left.\begin{matrix}3B\\-\frac{1}{3}\end{matrix}\right|x+\left.\begin{matrix}3C\\-B\end{matrix}\right|x^2+\left.\begin{matrix}3D\\-C\end{matrix}\right|x^3+\begin{matrix}\cdots.\\-\cdots.\end{matrix}$$

By Art. 3, we have

$$\begin{aligned}1=1,\quad 3B-\tfrac{1}{3}&=-1, \text{ whence } B=-\tfrac{2}{9};\\ 3C-B&=0, \text{ whence } C=-\tfrac{2}{27};\\ 3D-C&=0, \text{ whence } D=-\tfrac{2}{81};\\ &\text{etc.,} \qquad\qquad \text{etc.}\end{aligned}$$

Hence, $$\frac{1-x}{3x^2-x^3}=\tfrac{1}{3}x^{-2}-\tfrac{2}{9}x^{-1}-\tfrac{2}{27}-\tfrac{2}{81}x-\cdots.$$

EXERCISES I.

Expand the following fractions in series, to ascending powers of x, to four terms:

1. $\dfrac{1}{1-2x}$.
2. $\dfrac{3}{1+3x}$.
3. $\dfrac{6}{3-x}$.
4. $\dfrac{1+x}{1-x}$.
5. $\dfrac{2-5x}{1+2x}$.
6. $\dfrac{3x+x^2}{1-2x}$.
7. $\dfrac{x^3-3x^2}{x^2-2}$.
8. $\dfrac{1-x}{5x^2+2x^3}$
9. $\dfrac{1}{1+x+x^2}$.

10. $\dfrac{1+2x}{1+x-x^2}$. 11. $\dfrac{2-x}{1+2x-3x^2}$. 12. $\dfrac{3-2x^2}{2-3x+x^3}$.

13. $\dfrac{2+x-3x^2}{3-x+3x^2}$. 14. $\dfrac{x^4-3x^2+1}{1+x-2x^2}$. 15. $\dfrac{1}{2x^2-6x^3+x^4}$.

Expansion of Surds.

7. Ex. Expand $\sqrt{(1-x^2+2x^3)}$,

in a series, to ascending powers of x. Assume

$$\sqrt{(1-x^2+2x^3)}=1+Bx+Cx^2+Dx^3+Ex^4+\cdots.$$

Squaring both sides of the equation, we have

$$1-x^2+2x^3=1+2B\left|x+2C\right|x^2+2D\left|x^3+2E\right|x^4+\cdots,$$
$$\begin{array}{c|c|c|c} & +B^2 & +2BC & +2BD & \cdots. \\ & & & +C^2 & \cdots. \end{array}$$

Equating coefficients, $1=1$.

$$\begin{aligned} 2B&=0, &&\text{whence } B=0; \\ 2C+B^2&=-1, &&\text{whence } C=-\tfrac{1}{2}; \\ 2D+2BC&=2, &&\text{whence } D=+1; \\ 2E+2BD+C^2&=0, &&\text{whence } E=-\tfrac{1}{8};\text{ etc.} \end{aligned}$$

Hence $\sqrt{(1-x^2+2x^3)}=1-\frac{1}{2}x^2+x^3-\frac{1}{8}x^4+\cdots$

EXERCISES II.

Expand the following expressions in series, to ascending powers of x, to four terms:

1. $\sqrt{(1+x)}$. 2. $\sqrt{(a^2-2x^2)}$. 3. $\sqrt[3]{(1-x^3)}$.

4. $\sqrt{(4-2x+x^2)}$. 5. $\sqrt{(5+3x+9x^2)}$. 6. $\sqrt[3]{(1-x+x^2)}$.

Partial Fractions.

8. It is frequently desirable to separate a rational algebraical fraction into the simpler (*partial*) fractions of which it is the algebraical sum.

E.g., $$\frac{2x}{1-x^2}=\frac{1}{1-x}-\frac{1}{1+x}.$$

The process of separating a given fraction into its partial fractions is, therefore, the converse of addition (including subtraction) of fractions; and this fact must guide us in assuming the forms of the partial fractions.

We shall also assume that the degree of the numerator is at least one less than that of the denominator. A fraction whose numerator is of a degree equal to or greater than that of its denominator can be first reduced by division to the sum of an integral expression and a fraction satisfying the above condition. The latter fraction will then be decomposed.

The *denominators* of the partial fractions can be definitely assumed. For they are evidently those factors whose lowest common multiple is the denominator of the given fraction. But there is one case of doubt; namely, when a prime factor is repeated in the denominator of the given fraction.

E.g.,
$$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{3}{1-x}+\frac{2}{(1-x)^2}+\frac{1}{1+x};$$
$$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

We could not have decided, in advance, whether either of the two given fractions is the sum of two or of three partial fractions. There must necessarily be a partial fraction having $(1-x)^2$ as a denominator, since, otherwise, the L. C. M. of the denominators would not contain the prime factor $1-x$ to the second power. But it cannot be determined, in advance, whether there is a partial fraction having $1-x$ as a denominator.

In such cases, therefore, it is advisable to make provision for all possible partial fractions by assuming as denominators all repeated factors to the first power, second power, etc.

The numerators of partial fractions thereby assumed, which should not have been included, will acquire the value zero from the subsequent work, so that those fractions drop out of the result.

The *numerators* of the partial fractions must be assumed with undetermined coefficients. Since the numerator of the given fraction is, by the hypothesis, of degree at least one less than the denominator, the same must be true of each partial fraction. We therefore assume, for each numerator, a *complete* rational integral expression with undetermined coefficients of degree one lower than the corresponding denominator.

If any term in the assumed form of the numerator should not have been included, its coefficient will prove to be zero.

An exception to this principle occurs when the denominator of the partial fraction is the second or higher power of a prime factor, as, $(1-x)^2$. In that case the numerator is assumed as it would be according to the above principle if the prime factor occurred to the first power only.

We may briefly restate the above principles:

Separate the denominator of the given fraction into its prime factors. Assume as the denominator of a partial fraction each prime factor; in particular, when a prime factor enters to the nth power, assume that factor to the first power, second power, and so on, to the nth power, as a denominator.

Assume for each numerator a rational integral expression, with undetermined coefficients, of degree one lower than the prime factor in the corresponding denominator.

Let us first decompose the two fractions which we have used to illustrate the theory.

Ex. 1. $$\frac{6-2x^2}{(1-x)^2(1+x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.$$

Since the prime factor in the denominator of each partial fraction is of the first degree, each numerator is assumed to be of the zeroth degree.

Clearing the equation of fractions, we have

$$\begin{aligned}6-2x^2&=A(1-x)(1+x)+B(1+x)+C(1-x)^2\\&=(-A+C)x^2+(B-2C)x+A+B+C.\end{aligned}$$

Since this equation must be true for all values of x, we have

$$\left.\begin{aligned} -A+C&=-2,\\ B-2C&=0,\\ A+B+C&=6.\end{aligned}\right\}\text{ Whence } A=3,\ B=2,\ C=1.$$

Ex. 2. $$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.$$

The forms of the partial fractions are assumed the same as in Ex. 1. We have

$$3+x^2=(-A+C)x^2+(B-2C)x+A+B+C,$$

and then $$\left.\begin{aligned} -A+C&=1,\\ B-2C&=0,\\ A+B+C&=3.\end{aligned}\right\}\text{ Whence } A=0,\ B=2,\ C=1.$$

Therefore $$\frac{3+x^2}{(1-x)^2(1+x)}=\frac{2}{(1-x)^2}+\frac{1}{1+x}.$$

When the factors of the denominator of the given fraction are of the first degree, as in Exs. 1 and 2, the work may be shortened.

Begin with the equation

$$6-2x^2=A(1-x)(1+x)+B(1+x)+C(1-x)^2,$$

of Ex. 1. Since this equation is true for all values of x, we may substitute in it for x any value we please. Let us take such a value as will make one of the prime factors zero.

Substituting 1 for x, we obtain

$$4=2B,\text{ whence } B=2.$$

Next, letting $x=-1$, we have

$$4=4C,\text{ whence } C=1.$$

There is no other value of x which will make a prime factor zero, but any other value, the smaller the better, will give an equation in which we may substitute the values of B and C already obtained.

Letting $x=0$, we obtain

$$6=A+B+C, \text{ whence } A=3.$$

The same method can be applied to Ex. 2.

Ex. 3. $\dfrac{x^2-x+3}{x^3-1}=\dfrac{x^2-x+3}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}.$

In this example, the one prime factor being of the second degree, we assume the corresponding numerator to be a complete linear expression.

Clearing of fractions, we have

$$x^2-x+3=A(x^2+x+1)+(Bx+C)(x-1)=$$
$$(A+B)x^2+(A-B+C)x+A-C.$$

Equating coefficients of like powers of x, we obtain

$$A+B=1,\ A-B+C=-1,\ A-C=3;$$

whence, $\quad A=1,\ B=0,\ C=-2.$

Or, we might have used the second method, beginning with

$$x^2-x+3=A(x^2+x+1)+(Bx+C)(x-1).$$

Letting $x=1$, we obtain

$$3=3A, \text{ whence } A=1.$$

Since no other value of x will make a factor vanish, we take any simple values. When $x=0$, we have

$$3=A-C, \text{ whence } C=-2.$$

Finally, letting $x=-1$, we have

$$5=A+2B-2C, \text{ whence } B=0.$$

Therefore $\quad \dfrac{x^2-x+3}{x^3-1}=\dfrac{1}{x-1}-\dfrac{2}{x^2+x+1}.$

Ex. 4. $\dfrac{2-2x+4x^2}{(1+x^2)^2(1-x)}=\dfrac{Ax+B}{1+x^2}+\dfrac{Cx+D}{(1+x^2)^2}+\dfrac{E}{1-x}.$

The prime factors in the denominators of the first two partial fractions being of the second degree, expressions of the first degree are assumed as numerators.

Clearing of fractions, we have

$$\begin{aligned}
2-2x+4x^2 &= (Ax+B)(1+x^2)(1-x)+(Cx+D)(1-x)+E(1+x^2)^2 \\
&= (-A+E)x^4+(A-B)x^3+(-A+B-C+2E)x^2 \\
&\qquad +(A-B+C-D)x+(B+D+E).
\end{aligned}$$

Equating coefficients of like powers of x, we obtain

$$-A+E=0,\ A-B=0,\ -A+B-C+2E=4,$$
$$A-B+C-D=-2,\ B+D+E=2;$$

whence, $A=1,\ B=1,\ C=-2,\ D=0,\ E=1.$

Therefore $\dfrac{2-2x+4x^2}{(1+x^2)^2(1-x)}=\dfrac{x+1}{1+x^2}-\dfrac{2x}{(1+x^2)^2}+\dfrac{1}{1-x}.$

Ex. 5.

$$\frac{1}{(x+n)(x+n+1)(x+n+2)}=\frac{A}{x+n}+\frac{B}{x+n+1}+\frac{C}{x+n+2}.$$

Clearing of fractions, we have

$$1=A(x+n+1)(x+n+2)+B(x+n)(x+n+2)$$
$$+C(x+n)(x+n+1).$$

Letting $x=-n$, we have $1=2A$, or $A=\frac{1}{2}$;

$x=-n-1$, $1=-B$, or $B=-1$;

$x=-n-2$, $1=2C$, or $C=\frac{1}{2}$.

Therefore

$$\frac{1}{(x+n)(x+n+1)(x+n+2)}=\frac{1}{2(x+n)}-\frac{1}{x+n+1}+\frac{1}{2(x+n+2)}.$$

9. The General Term. — The following examples illustrate the method of finding the general term of the expansion of a rational fraction in a series, to ascending powers of x.

Ex. 1. Find the general term of the expansion of $\dfrac{2+7x}{1+x-2x^2}$.

We have

$$\begin{aligned}
\frac{2+7x}{1+x-2x^2} &= \frac{3}{1-x}-\frac{1}{1+2x} \\
&= 3(1+x+x^2+\cdots+x^n+\cdots) \\
&\quad -[1+(-2x)+(-2x)^2+\cdots+(-2x)^n+\cdots].
\end{aligned}$$

The expansions of the above partial fractions, and similar ones, are readily obtained by the formula

$$\frac{1}{1-x}=1+x+x^2+\cdots.$$

The required general term is the sum of the $(n+1)$th terms of the above expansions.

We have $3x^n-(-2x)^n=x^n[3+(-1)^{n+1}2^n]$.

The expansion of the given fraction can be obtained from this general term. Giving to n the values 0, 1, 2, 3, ⋯, we obtain

$$\frac{2+7x}{1+x-2x^2}=2+5x-x^2+11x^3-\cdots+[3+(-1)^{n+1}2^n]x^n+\cdots.$$

Ex. 2. Find the general term of the expansion of

$$\frac{10-7x+6x^2}{(2-x)(1+x^2)}.$$

We have

$$\begin{aligned}\frac{10-7x+6x^2}{(2-x)(1+x^2)}&=\frac{4}{2-x}+\frac{3-2x}{1+x^2}=\frac{2}{1-\frac{1}{2}x}+\frac{3-2x}{1+x^2}\\&=2[1+\tfrac{1}{2}x+(\tfrac{1}{2}x)^2+\cdots+(\tfrac{1}{2}x)^{2n}+(\tfrac{1}{2}x)^{2n+1}+\cdots]\\&\quad+(3-2x)[1+(-x^2)+(-x^2)^2+\cdots+(-x^2)^n+\cdots]\\&=2[1+\tfrac{1}{2}x+\tfrac{1}{4}x^2+\cdots+(\tfrac{1}{2})^{2n}x^{2n}+(\tfrac{1}{2})^{2n+1}x^{2n+1}+\cdots]\\&\quad+[3-3x^2+3x^4-\cdots+(-1)^n3x^{2n}+\cdots]\\&\quad+[-2x+2x^3-2x^5+\cdots+(-1)^{n+1}2x^{2n+1}+\cdots].\end{aligned}$$

Observe that it is necessary to distinguish between even and odd powers of x.

Terms containing *even* powers of x are obtained from

$$(\tfrac{1}{2})^{2n-1}x^{2n}+(-1)^n3x^{2n},\ =x^{2n}[(\tfrac{1}{2})^{2n-1}+3(-1)^n];$$

and terms containing *odd* powers from

$$(\tfrac{1}{2})^{2n}x^{2n+1}+(-1)^{n+1}2x^{2n+1},\ =x^{2n+1}[(\tfrac{1}{2})^{2n}+2(-1)^{n+1}].$$

The expansion is readily obtained from these general terms.

EXERCISES III.

Separate the following fractions into partial fractions:

1. $\dfrac{6}{(x-2)(1-2x)}$. **2.** $\dfrac{7}{(5+3x)(x+4)}$.

3. $\dfrac{3x-1}{(x+3)(x-2)}$. **4.** $\dfrac{1-x}{(3x+2)(x+1)}$.

5. $\dfrac{5}{1-x^2}$. **6.** $\dfrac{6x}{x^2-4}$. **7.** $\dfrac{1+x}{9-x^2}$.

8. $\dfrac{1}{7x-x^2-12}$. **9.** $\dfrac{x^2+2x-1}{9x^2-16}$.

10. $\dfrac{3x+2}{(x^2-1)(x-2)}$. **11.** $\dfrac{x^2+90x-9}{6(x^2-9)(x-3)}$.

12. $\dfrac{3x^2+1}{(x+1)(x-1)^2}$. **13.** $\dfrac{x^2+5x+10}{(x+1)(x+2)(x+3)}$.

14. $\dfrac{5x(x+3)}{(2x+1)(2x-1)(x+1)}$. **15.** $\dfrac{3-x}{(2x+1)(2x+3)(x-1)}$.

16. $\dfrac{x}{(x-1)^3}$. **17.** $\dfrac{1}{x^3-1}$. **18.** $\dfrac{2}{x^3+1}$.

19. $\dfrac{x+1}{x^3-1}$. **20.** $\dfrac{1}{x^4-1}$. **21.** $\dfrac{1}{x^2(x^2+1)}$.

22–28. Find the general terms of the expansions, to ascending powers of x, of the fractions in Exx. 5–11.

Find the general term of the expansions of the following fractions, to ascending powers of x:

29. $\dfrac{1}{2x(x^2+1)}$. **30.** $\dfrac{5x^2-6x-13}{10(x+3)(1+x^2)}$. **31.** $\dfrac{6x+26}{3(x-4)(2+3x^2)}$.

Reversion of Series.

10. If one variable be equal to a series of positive integral ascending powers of a second variable, the second variable can be expressed in a series of positive integral ascending powers of the first. This process is called *reversion of series.*

Ex. 1. Revert the series

$$y = x + 2x^2 + 3x^3 + \cdots.$$

Assume $\qquad x = Ay + By^2 + Cy^3 + \cdots, \qquad (1)$

and substitute in the second member of the last equation the value of y given by the first. Then

$$\begin{aligned} x &= A(x + 2x^2 + 3x^3 + \cdots) + B(x + 2x^2 + 3x^3 + \cdots)^2 \\ &\quad + C(x + 2x^2 + 3x^3 + \cdots)^3 + \cdots \\ &= Ax + \left.\begin{matrix} 2A \\ +\ B \end{matrix}\right| x^2 + \left.\begin{matrix} 3A \\ +4B \\ +\ C \end{matrix}\right| x^3 + \cdots. \end{aligned}$$

Hence $\qquad A = 1.$

$$\begin{aligned} 2A + B &= 0, \text{ whence } B = -2; \\ 3A + 4B + C &= 0, \text{ whence } C = \ \ 5; \\ \text{etc.,} & \qquad\qquad \text{etc.} \end{aligned}$$

Substituting these values of A, B, C, $\cdots$, in (1), we have

$$x = y - 2y^2 + 5y^3 + \cdots.$$

If the series for y in terms of x contain a term free from x, we must find a value of x in a series of powers of y minus that term.

Ex. 2. Revert the series

$$y = 1 + x + x^2 + x^3 + \cdots,$$

or $\qquad y - 1 = x + x^2 + x^3 + \cdots. \qquad (2)$

Assuming $\quad x = A(y-1) + B(y-1)^2 + \cdots,$

and proceeding as in Ex. 1, we obtain $A = 1$, $B = -1$, $C = 1$.

Therefore $x = (y-1) - (y-1)^2 + (y-1)^3 - \cdots.$

EXERCISES IV.

Revert each of the following series to four terms:

1. $y = x + x^2 + x^3 + \cdots.$
2. $y = x + 3x^2 + 5x^3 + \cdots.$
3. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots.$
4. $y = 1 - x + 2x^2 - \cdots.$
5. $y = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \cdots.$
6. $y = ax + bx^2 + cx^3 + \cdots.$

CHAPTER XXVII.

THE BINOMIAL THEOREM.

1. In Ch. XXII. it was proved by induction that, when n is a positive integer,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

We will here give a briefer proof, based upon the theory of combinations.

Consider the following continued product of n factors:

$$n \text{ factors} \begin{cases} a+b \\ a+b \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ a+b \end{cases}$$

The first term of the product is formed by taking an a from each factor, giving a^n. A second term is formed by taking an a from $n-1$ factors and a b from the remaining factor, giving $a^{n-1}b$. But such a term can be formed in as many ways as one b can be taken from n b's, *i.e.*, in ${}_nC_1$ ways. Therefore the product so far is $a^n + {}_nC_1a^{n-1}b$.

A third term is formed by taking an a from $n-2$ factors and a b from the remaining two factors, giving $a^{n-2}b^2$. But such a term can be formed in as many ways as two b's can be taken from n b's, *i.e.*, in ${}_nC_2$ ways. Consequently, the product to this point is $a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2$.

In general, an a can be taken from each of $n-k+1$ factors and a b from each of the remaining $k-1$ factors, giving $a^{n-k+1}b^{k-1}$. But such a term can evidently be formed in ${}_nC_{k-1}$ ways.

We thus obtain

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \cdots + {}_nC_{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

But $\quad {}_nC_1 = \binom{n}{1},\ {}_nC_2 = \binom{n}{2},\ {}_nC_3 = \binom{n}{3},\ \cdots,\ {}_nC_{k-1} = \binom{n}{k-1}.$

Therefore, $\quad (a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3$

$$+ \cdots + \binom{n}{k-1}a^{n-k+1}b^{k-1} + \cdots.$$

Properties of Binomial Coefficients.

2. The kth term, counting from the beginning of the expansion, contains b^{k-1}, and is ${}_nC_{k-1}a^{n-k+1}b^{k-1}$. The kth term, counting from the end, contains a^{k-1}, and therefore b^{n-k+1}, and is ${}_nC_{n-k+1}a^{k-1}b^{n-k+1}$.

But, by Ch. XXIII., Art. 14, ${}_nC_{k-1} = {}_nC_{n-k+1}$. We therefore conclude:

In the expansion of $(a+b)^n$, *wherein* n *is a positive integer, the coefficients of terms equally distant from the beginning and end of the expansion are equal.*

3. By Art. 1, the coefficient of the $(k+1)$th term is ${}_nC_k$. Therefore, by Ch. XXIII., Art. 15, we have:

The greatest binomial coefficient, when n *is even, is* ${}_nC_{\frac{n}{2}}$; *and when* n *is odd, is* ${}_nC_{\frac{n-1}{2}} = {}_nC_{\frac{n+1}{2}}$.

4. In $\quad (1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n, \quad$ let $\quad x = 1.$

Then $$2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n.$$

That is, *the sum of the binomial coefficients is* 2^n.

5. From Art. 4, we have

$${}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1.$$

That is, *the total number of combinations of* n *things, taken one at a time, two at a time, and so on, to* n *at a time, is* $2^n - 1$.

6. In $(1+x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n$, let $x = -1$.

Then $$1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots = 0,$$

or $$1 + {}_nC_2 + {}_nC_4 + \cdots = {}_nC_1 + {}_nC_3 + \cdots.$$

That is, *in the binomial expansion, the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

Binomial Theorem for Any Rational Exponent.

7. From Ch. XXII., Art. 4, we have

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots, \qquad (1)$$

when n is a positive integer. In this case the expansion ends with the $(n+1)$th term, since the coefficients of the $(n+2)$th and all succeeding terms contain $n - n$, or 0, as a factor. But if n be not a positive integer, the expression on the right of (1) will continue without end, since no factor of the form $n - k + 1$ can reduce to 0. Therefore this series will have no meaning unless it be convergent.

8. In Chap. XXV., Art. 27, it was proved that the series

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

is convergent when x lies between -1 and $+1$. It remains to be proved, therefore, that in this case the above series represents $(1+x)^n$, when n is a fraction or negative.

9. Since the reasoning will turn upon the value of n, we shall call the expression

$$1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

a function of n, and abbreviate it by $f(n)$, *for all rational values of n.* To understand the following reasoning, the

student should notice that for all positive integral values of n, $(1+x)^n = f(n)$, as, $(1+x)^3 = f(3)$; and that it remains to be proved that $(1+x)^n = f(n)$, when n is a fraction or negative; as, for example, that $(1+x)^{\frac{2}{3}} = f(\frac{2}{3})$.

10. We now have

$$f(m) = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{k-1}x^{k-1} + \cdot$$

$$f(n) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k-1}x^{k-1} + \cdots$$

for all values of x between -1 and $+1$.

We will assume that the product $f(m) \times f(n)$ is a convergent series, when the two series are convergent. The proof of this principle is beyond the scope of this book. We then have

$$\begin{aligned} f(m) \times f(n) = 1 &+ \left[\binom{m}{1} + \binom{n}{1}\right]x + \left[\binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2}\right]x^2 + \cdots \\ &+ \left[\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \binom{m}{k-3}\binom{n}{2} + \cdots \right. \\ &\left. + \binom{m}{2}\binom{n}{k-3} + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1}\right]x^{k-1} + \cdots. \end{aligned}$$

But, by Ch. XXIII., Art. 17,

$$\binom{m}{1} + \binom{n}{1} = \binom{m+n}{1}, \quad \binom{m}{2} + \binom{m}{1}\binom{n}{1} + \binom{n}{2} = \binom{m+n}{2},$$

$$\binom{m}{k-1} + \binom{m}{k-2}\binom{n}{1} + \cdots + \binom{m}{1}\binom{n}{k-2} + \binom{n}{k-1} = \binom{m+n}{k-1};$$

therefore $$f(m) \times f(n) = f(m+n), \qquad (1)$$

for all rational values of m and n.

Then $f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p)$.

In general,

$$f(m) \times f(n) \times f(p) \times \cdots \times f(r) = f(m+n+p+\cdots+r), \qquad (2)$$

for all rational values of $m, n, p, \cdots, r$.

11. Fractional Exponents. — Let

$$m = n = p = \cdots = r = \frac{u}{v},$$

wherein u and v are positive integers. Taking v factors, we now have

$$f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times f\left(\frac{u}{v}\right) \times \cdots v \text{ factors} = f\left(\frac{u}{v} + \frac{u}{v} + \frac{u}{v} + \cdots v \text{ summands}\right),$$

or

$$\left[f\left(\frac{u}{v}\right)\right]^v = f\left(\frac{u}{v} \cdot v\right) = f(u).$$

Now, since u is a positive integer, $(1 + x)^u = f(u)$.

Therefore $(1 + x)^u = \left[f\left(\frac{u}{v}\right)\right]^v$, or $(1 + x)^{\frac{u}{v}} = f\left(\frac{u}{v}\right)$.

That is, $$(1 + x)^{\frac{u}{v}} = 1 + \left\{\begin{matrix}\frac{u}{v} \\ 1\end{matrix}\right\} x + \left\{\begin{matrix}\frac{u}{v} \\ 2\end{matrix}\right\} x^2 + \cdots.$$

12. Negative Exponents, Integral or Fractional. — In (1), Art. 10, let $$m = -n.$$

We then have $f(-n) \times f(n) = f(n - n) = f(0) = 1$,

since $$f(0) = 1 + 0 \cdot x + \cdots = 1.$$

Therefore $$\frac{1}{f(n)} = f(-n). \qquad (1)$$

Since n is a positive integer or fraction, $(1 + x)^n = f(n)$, and (1) becomes

$$\frac{1}{(1 + x)^n} = f(-n), \text{ or } (1 + x)^{-n} = f(-n).$$

That is, $(1 + x)^{-n} = 1 + \binom{-n}{1} x + \binom{-n}{2} x^2 + \cdots.$

13. Expansion of $(a + b)^n$. — We have

$$(a + b)^n = \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n \left(1 + \frac{b}{a}\right)^n, \qquad (1)$$

and

$$(a + b)^n = \left[b\left(1 + \frac{a}{b}\right)\right]^n = b^n \left(1 + \frac{a}{b}\right)^n. \qquad (2)$$

When b is numerically less than a,

$$\left(1+\frac{b}{a}\right)^n = 1 + \binom{n}{1}\frac{b}{a} + \binom{n}{2}\frac{b^2}{a^2} + \cdots,$$

and, by (1) above,

$$(a+b)^n = a^n\left[1 + \binom{n}{1}\frac{b}{a} + \binom{n}{2}\frac{b^2}{a^2} + \cdots\right]$$

$$= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots. \quad (3)$$

In a similar way it can be shown that, when a is numerically less than b,

$$(a+b)^n = b^n + \binom{n}{1}b^{n-1}a + \binom{n}{2}b^{n-2}a^2 + \cdots. \quad (4)$$

Notice that when n is a fraction or negative, formula (3) or (4) must be used according as a is numerically greater or less than b.

14. Ex. Expand $\dfrac{1}{\sqrt[3]{(a-4b^2)}}$ to four terms.

If we assume $a > 4b^2$, we have, by (3), Art. 13,

$$\frac{1}{\sqrt[3]{(a-4b^2)}} = (a-4b^2)^{-\frac{1}{3}} = a^{-\frac{1}{3}} + (-\tfrac{1}{3})a^{-\frac{4}{3}}(-4b^2)$$

$$+ \frac{-\frac{1}{3}(-\frac{4}{3})}{1\cdot 2}a^{-\frac{7}{3}}(-4b^2)^2$$

$$+ \frac{-\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})}{1\cdot 2\cdot 3}a^{-\frac{10}{3}}(-4b^2)^3 + \cdots$$

$$= \frac{1}{\sqrt[3]{a}} + \frac{4b^2}{3a\sqrt[3]{a}} + \frac{32b^4}{9a^2\sqrt[3]{a}} + \frac{896b^6}{81a^3\sqrt[3]{a}} + \cdots.$$

If $a < 4b^2$, we should have used (4), Art. 13.

Any particular term can be written as in Ch. XXII., Art. 9.

15. Extraction of Roots of Numbers. — Ex. Find $\sqrt{17}$ to four decimal places. We have

$$\begin{aligned}\sqrt{17} &= \sqrt{(16+1)} = 4\left(1+\tfrac{1}{16}\right)^{\frac{1}{2}}\\ &= 4\left[1+\frac{1}{2}\times\frac{1}{16}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}\left(\frac{1}{16}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3}\left(\frac{1}{16}\right)^3+\cdots\right]\\ &= 4\,(1+.03125-.00049+.00002-\cdots)\\ &= 4\times 1.03078 = 4.12312.\end{aligned}$$

Therefore $\sqrt{17} = 4.1231$, to four decimal places.

EXERCISES.

Expand to four terms:

1. $(1+a)^{\frac{1}{2}}$.
2. $(1-x)^{-1}$.
3. $(1-x)^{-3}$.
4. $(1+x^2)^{\frac{3}{2}}$.
5. $(1+x)^{-4}$.
6. $(1-y^2)^{-2}$.
7. $(x^2+y)^{-\frac{2}{3}}$.
8. $(x-y^2)^{-4}$.
9. $(27+5\,x)^{\frac{2}{3}}$.
10. $(8\,a^3-3\,b)^{\frac{1}{3}}$.
11. $(3+2\,x)^{\frac{3}{4}}$.
12. $(5\,a^2-3\,b^3)^{-\frac{2}{3}}$.
13. $\dfrac{1}{\sqrt{(a^2-b^2)}}$.
14. $\dfrac{1}{\sqrt[3]{(a^3-b)}}$.
15. $\dfrac{1}{\sqrt{(2\,x^{-1}-34^{\frac{1}{2}})^3}}$.

Find the

16. 4th term of $(1-2\,x)^{\frac{1}{3}}$.
17. 6th term of $(1+a^2b^{-\frac{1}{3}})^{-3}$.
18. 5th term of $(x^{\frac{2}{3}}-x^{-1}y^2)^{-\frac{3}{4}}$.
19. 8th term of $(a^3\sqrt{b}-2\,b\sqrt[3]{a})^{-\frac{1}{2}}$.
20. k—5th term of $(1+x^{\frac{1}{3}}y^{\frac{1}{2}})^{-2}$.
21. 2 kth term of $[x^2-\sqrt{(xy)}]^{\frac{2}{3}}$.

Find to four places of decimals the values of:

22. $\sqrt{5}$.
23. $\sqrt{27}$.
24. $\sqrt[3]{35}$.
25. $\sqrt[4]{700}$.
26. $\sqrt[5]{258}$.
27. Find the term in $(3\,x^3-x^2y)^{\frac{5}{3}}$ containing x^2.
28. Find the term in $\left(a+\dfrac{1}{2\sqrt{a}}\right)^{-\frac{1}{2}}$ containing a^{-11}.

CHAPTER XXVIII.

LOGARITHMS.

1. A value of x can always be found to satisfy an equation of the form

$$10^x = n,$$

wherein n is any real positive number. *E.g.*, when $n = 10$, $x = 1$, when $n = 100$, $x = 2$, when $n = 1000$, $x = 3$, etc.

The proof of this principle is beyond the scope of this book.

When n is not an integral power of 10, the value of x is irrational, and can be expressed only approximately. Thus, when $n = 24$, the corresponding value of x has been found to be $1.38021\cdots$, to five decimal places; or

$$10^{1.38021\cdots} = 24.$$

A value of x is called the *logarithm* of the corresponding value of n, and *10* is called the *base.*

In general, a value of x which satisfies the equation $b^x = n$, is called *the logarithm of n to the base b.*

E.g., since $2^3 = 8$, 3 is the logarithm of 8 to the base 2; since $10^2 = 100$, 2 is the logarithm of 100 to the base 10.

The Logarithm of a given number n to a given base b is, therefore, the exponent of the power to which the base b must be raised to produce the number n.

2. The relation $b^x = a$ is also written $x = \log_b a$, read *x is the logarithm of a to the base b.* Thus,

$$2^3 = 8 \quad \text{and } 3 = \log_2 8,$$

$$10^2 = 100 \text{ and } 2 = \log_{10} 100,$$

are equivalent ways of expressing one and the same relation.

3. The theory of logarithms is based upon the idea of representing all positive numbers, in their natural order, as powers of one and the same base.

Thus, 4, 8, 16, 32, 64, etc., can all be expressed as powers of a common base 2; as $4=2^2$, $8=2^3$, $16=2^4$, etc. Since, also, all the numbers intermediate between those given above can be expressed as powers of 2, the exponents of these powers are the logarithms of the corresponding numbers.

The logarithms of all positive numbers to a given base form what is called a **System of Logarithms**. The base is then called the *base of the system.*

It follows from Art. 1, that any positive number except 1 may be taken as the base of a system of logarithms.

EXERCISES I.

Express the following relations in the language of logarithms:

1. $5^2=25$. **2.** $2^5=32$. **3.** $7^3=343$. **4.** $3^7=2187$.

Express the following relations in terms of powers:

5. $\log_3 81=4$. **6.** $\log_9 81=2$. **7.** $\log_4 64=3$. **8.** $\log_2 64=6$.

Determine the values of the following logarithms:

9. $\log_2 32$. **10.** $\log_{\frac{1}{2}} 128$. **11.** $\log_2 .5$. **12.** $\log_2 .25$.

13. $\log_4 64$. **14.** $\log_{64} 8$. **15.** $\log_2 .125$. **16.** $\log_5 .04$.

To the base 16, what numbers have the following logarithms?

17. 0. **18.** $\frac{1}{2}$. **19.** -2. **20.** $\frac{3}{2}$. **21.** $-\frac{1}{4}$.

Principles of Logarithms.

4. *The logarithm of* 1 *to any base is* 0. For $b^0=1$, or $\log_b 1=0$.

5. *The logarithm of the base itself is* 1. For $b^1=b$, or $\log_b b=1$.

6. *The logarithm of a product is equal to the sum of the logarithms of its factors;* or,

$$\log_b (m \times n)=\log_b m+\log_b n.$$

Let $\qquad \log_b m = x$ and $\log_b n = y$;

then $\quad b^x = m$ and $b^y = n$, and therefore, $mn = b^x b^y = b^{x+y}$.

Translated into the language of logarithms, this result reads

$$\log_b(mn) = x + y.$$

But $\qquad x = \log_b m$ and $y = \log_b n$,

and consequently

$$\log_b(mn) = \log_b m + \log_b n,$$

for all positive values of b.

This result may be readily extended to a product of any number of factors. For,

$$\log_b(mnp) = \log_b(mn) + \log_b p = \log_b m + \log_b n + \log_b p.$$

And, in like manner, for any number of factors.

E.g. Given $\log_2 32 = 5$, and $\log_2 64 = 6$; what is the logarithm of 2048 to the base 2?

Since $2048 = 32 \cdot 64$, we have

$$\log_2 2048 = \log_2 32 + \log_2 64 = 5 + 6 = 11$$

7. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor;* or,

$$\log_b (m \div n) = \log_b m - \log_b n.$$

Let $\qquad \log_b m = x$ and $\log_b n = y$;

then $b^x = m$ and $b^y = n$, and therefore $m \div n = b^x \div b^y = b^{x-y}$.

In the language of logarithms the last equation is

$$\log_b (m \div n) = x - y = \log_b m - \log_b n,$$

for all positive values of b.

E.g. Given $\log_3 3 = 1$ and $\log_3 2187 = 7$, what is the logarithm of 729 to the base 3?

Since $\qquad 729 = \frac{2187}{3}$,

we have $\quad \log_3 729 = \log_3 2187 - \log_3 3 = 7 - 1 = 6.$

8. Both m and n may be products, or the quotient of two numbers.

E.g., $$\log_{10}\frac{4\times 5}{9\times 8}=\log_{10}(4\times 5)-\log_{10}(9\times 8)$$
$$=\log_{10}4+\log_{10}5-\log_{10}9-\log_{10}8.$$

9. *The logarithm of the reciprocal of any number is the opposite of the logarithm of the number.*

For, $$\log_b\frac{1}{n}=\log_b 1-\log_b n$$
$$=-\log_b n, \text{ since } \log_b 1=0.$$

E.g., $$\log_2 4=2, \text{ and } \log_2\tfrac{1}{4}=-2.$$

10. *The logarithm of any power, integral or fractional, of a number is equal to the logarithm of the number multiplied by the exponent of the power;* or

$$\mathbf{\log m^p = p\log m.}$$

Let $$\log_b m=x, \text{ then } b^x=m.$$

Raising both sides of the last equation to the pth power, we have $b^{px}=m^p$, or $\log_b(m^p)=px=p\log_b m$.

E.g., If $\log_5 25=2$, what is $\log_5 25^3$?

We have $\log_5 25^3=3\log_5 25=3\times 2=6.$

11. When the exponent is a positive fraction whose numerator is 1, this principle may be conveniently stated thus:

The logarithm of a root of a number is the logarithm of the number divided by the index of the root.

For, $$\log_b(m^{\frac{1}{q}})=\frac{1}{q}\log m=\frac{\log_b m}{q}.$$

E.g., If $\log_7 2401=4$, what is $\log_7\sqrt{2401}$?

We have $$\log_7\sqrt{2401}=\frac{1}{2}\log_7 2401=\frac{1}{2}\cdot 4=2.$$

EXERCISES II.

Express the following logarithms in terms of $\log a$, $\log b$, $\log c$, and $\log d$:

1. $\log \dfrac{abc}{d}$. **2.** $\log \dfrac{d}{abc}$. **3.** $\log \dfrac{ac^2}{bd^2}$. **4.** $\log \left(\dfrac{ac}{bd}\right)^2$.

5. $\log a^{\frac{5}{6}} d^{-\frac{2}{3}} \sqrt{b} \sqrt{c}$. **6.** $\log \dfrac{2\,ab^2}{3\,c\sqrt{d}}$. **7.** $\log \dfrac{a^{-2} b^{\frac{3}{2}}}{\sqrt{(c^5 d^{-3})}}$.

Express the following sums of logarithms as logarithms of products and quotients.

8. $\log a + \log b - \log c$. **9.** $\log a - (\log b + \log c)$.

10. $3 \log a - \frac{1}{2} \log (b + c)$. **11.** $\frac{1}{2} \log (1 - x) + \frac{3}{2} \log (1 + x)$.

12. $2 \log \dfrac{a}{b} + 3 \log \dfrac{b}{a}$. **13.** $2 \log a - \frac{2}{3} \log b + \frac{1}{2} \log c$.

Given $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, $\log_{10} 5 = .69897$, $\log_{10} 7 = .84510$, find the values of the following logarithms, to the base 10:

14. $\log 50$. **15.** $\log 6$. **16.** $\log 8$. **17.** $\log 9$.

18. $\log 12$. **19.** $\log 36$. **20.** $\log 108$. **21.** $\log 4\frac{1}{2}$.

22. $\log 2\frac{2}{3}$. **23.** $\log 5\frac{5}{6}$. **24.** $\log 5\frac{1}{7}$. **25.** $\log 360$.

26. $\log 3072$. **27.** $\log 3500$. **28.** $\log 5880$.

29. $\log \sqrt{72}$. **30.** $\log \sqrt{180}$. **31.** $\log \sqrt{1715}$.

32. $\log \dfrac{\sqrt[5]{490}}{\sqrt[6]{96}}$. **33.** $\log \dfrac{\sqrt[6]{9\frac{3}{5}} \times \sqrt{105}}{\sqrt[3]{72} \times \sqrt[4]{8\frac{2}{5}}}$. **34.** $\log \dfrac{(4\frac{2}{3})^3}{(11\frac{2}{3})^{\frac{3}{2}}}$.

Systems of Logarithms.

12. The two most important systems of logarithms are:

(i.) The system whose base is 10. This system was introduced, in 1615, by the Englishman, Henry Briggs.

Logarithms to the base 10 are called **Common**, or **Briggs's Logarithms**.

(ii.) The system whose base is the sum of the following infinite series,

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots.$$

The value of this sum, which to seven places of decimals is 2.7182818, is denoted by the letter e.

Logarithms to the base e are called **Natural Logarithms**; sometimes also **Napierian Logarithms**, in honor of the inventor of logarithms, the Scotch Baron Napier, a contemporary of Briggs. Napier himself did not, however, introduce this system of logarithms.

These two systems are the only ones which have been generally adopted; the common system is used in practical calculations, the natural system in theoretical investigations. The reason that in all practical calculations the common system of logarithms is superior to other systems is because its base 10 is also the base of our decimal system of numeration.

The logarithms of most numbers are irrational, and thus approximate values are used.

Properties of Common Logarithms.

13. In the following articles the subscript denoting the base 10 will be omitted.

We now have

$$(a)\begin{cases} 10^0 = 1, \text{ or } \log 1 = 0; \\ 10^1 = 10, \text{ or } \log 10 = 1; \\ 10^2 = 100, \text{ or } \log 100 = 2; \\ 10^3 = 1000, \text{ or } \log 1000 = 3; \end{cases}$$

$$\cdots\cdots\cdots\cdots$$

$$(b)\begin{cases} 10^{-1} = .1, \text{ or } \log .1 = -1; \\ 10^{-2} = .01, \text{ or } \log .01 = -2; \\ 10^{-3} = .001, \text{ or } \log .001 = -3; \\ 10^{-4} = .0001, \text{ or } \log .0001 = -4; \end{cases}$$

$$\cdots\cdots\cdots\cdots$$

Evidently the logarithms of all positive numbers, except positive and negative integral powers of 10, consist of an integral and a decimal part. Thus, since $10^1 < 85 < 10^2$, we have $1 < \log 85 < 2$, or $\log 85 = 1 +$ *a decimal.*

14. The integral part of a logarithm is called its **Characteristic.**

The decimal part of a logarithm is called its **Mantissa.**

15. Since a number having one digit in its integral part, as 7.3, lies between 10^0 and 10^1, it follows from table (a) that its logarithm lies between 0 and 1, *i.e.*, is $0 +$ *a decimal.* Since any number having two digits in its integral part, as 76.4, lies between 10^1 and 10^2, its logarithm lies between 1 and 2, that is, is $1 +$ *a decimal.* In general, since any number having n digits in its integral part lies between 10^{n-1} and 10^n, its logarithm lies between $n-1$ and n, *i.e.*, is $n - 1 +$ *a decimal.* We therefore have:

(i.) *The characteristic of the logarithm of a number greater than unity is positive, and is one less than the number of digits in its integral part.*

E.g., $\log 2756.3 = 3 +$ *a decimal.*

Since a number less than 1 having no cipher immediately following the decimal point lies between 10^0 and 10^{-1}, it follows from table (b) that its logarithm lies between 0 and -1, *i.e.*, is $-1 +$ *a positive decimal.* Since a number less than 1 having one cipher immediately following the decimal point lies between 10^{-1} and 10^{-2}, its logarithm lies between -1 and -2, *i.e.*, is $-2 +$ *a positive decimal.* In general, since a number less than 1 having n ciphers immediately following the decimal point lies between 10^{-n} and $10^{-(n+1)}$, its logarithm lies between $-n$ and $-(n+1)$, *i.e.*, is $-(n+1) +$ *a positive decimal.* We therefore have:

(ii.) *The characteristic of the logarithm of a number less than 1 is negative, and is numerically one greater than the number of ciphers immediately following the decimal point.*

E.g., $\log .00035 = -4 +$ *a positive decimal.*

It follows conversely from (i.) and (ii.):

(iii.) *If the characteristic of a logarithm be* $+n$, *there are* $n+1$ *digits in the integral part of the corresponding number.*

(iv.) *If the characteristic of a logarithm be* $-n$, *there are* $n-1$ *ciphers immediately following the decimal point of the corresponding number.*

16. It has been found that $538 = 10^{2.73078}$ to five decimal places, or $\log 538 = 2.73078$. We also have

$$\log .0538 = \log \tfrac{538}{10000} = \log 538 - \log 10000 = 2.73078 - 4$$
$$= .73078 - 2;$$
$$\log 5.38 = \log \tfrac{538}{100} = \log 538 - \log 100 = 2.73078 - 2$$
$$= .73078;$$
$$\log 53800 = \log (538 \times 100) = \log 538 + \log 100$$
$$= 2.73078 + 2 = 4.73078.$$

These examples illustrate the following principle:

If two numbers differ only in the position of their decimal points, their logarithms have different characteristics but the same positive mantissa.

17. The characteristic and the mantissa of a number less than 1 may be connected by the decimal point, if the sign $(-)$ be written over the characteristic to indicate that the characteristic only is negative, and not the entire number.

Thus, instead of $\log .00709 = .85065 - 3 = -3 + .85065$, we may write $\bar{3}.85065$; this must be distinguished from the expression -3.85065, in which the integer and the decimal are both negative. Similarly,

$$\log .082 = \bar{2}.91381, \text{ while } \log 820 = 2.91381.$$

Five-Place Table of Logarithms.

18. The logarithms, to the base 10, of a set of consecutive integers have been computed.

In tabulating these logarithms, compactness is important.

For this reason, all unnecessary detail is omitted. Since the characteristic of the logarithm of any number can, as we have seen, be determined by inspection, it is unnecessary to write it with the mantissa in the table. Consequently, only the mantissas, *without the decimal points,* are there given.

Neither is it necessary to give the logarithms of decimal fractions, since their mantissas are the same as the mantissas of the numbers obtained by omitting the decimal point.

The logarithms may be carried to any number of decimal places, and the extent to which they are carried depends upon the degree of accuracy required in their use.

19. The accompanying five-place table gives the mantissas of the logarithms of all consecutive integers from 1 to 9999 inclusive.

In this table the first three figures of each number are given in the column headed N, and the fourth figure in the horizontal line over the table. The first figure, which is the same for all numbers in a given column, is printed in every tenth number only.

The columns headed 0, 1, 2, 3, etc., contain the mantissas, with decimal points omitted.

In the column headed 0, when the first two figures are not printed, they are to be taken from the last mantissa above which is printed in full.

In the columns headed 1, 2, 3, etc., the last three figures only are printed; the first two are to be taken from the column headed 0 in the same horizontal line.

When a star is prefixed to the last three figures of a mantissa, the first two figures are to be taken from the line below.

To Find the Logarithm of a Given Number.

20. When the Number consists of Four or Fewer Figures. — Take the mantissa that is in the horizontal line with the first three figures and in the column under the fourth figure of the given number

Determine the characteristic by Art. 15.

E.g., $\log 2583 = 3.41212$, $\log 46.32 = 1.66577$.

In writing logarithms with negative characteristics it is customary to modify the characteristics so that 10 is uniformly subtracted from the logarithms.

Thus, $\bar{2}.45926 = .45926 - 2 = 8.45926 - 10$;

$\bar{4}.37062 = .37062 - 4 = 6.37062 - 10$.

That is, we add 10 to the negative characteristic, and write -10 after the logarithm.

$$\log .5757 = 9.76020 - 10, \log .02768 = 8.44217 - 10.$$

Observe that the first two figures of the mantissa of log .5757 are taken from the line below, in accordance with the directions in Art. 19.

If the given number consists of fewer than four figures, annex ciphers until it has four figures, in taking the mantissa from the table.

E.g., mantissa of log 78 = mantissa of log 7800 = .89209,

and $\log 78 = 1.89209$.

In like manner,

$$\log 583 = 2.76567, \log .02 = 8.30103 - 10.$$

21. When the Number consists of more than Four Significant Figures. — The method used is called *interpolation*, and depends upon the following property of logarithms:

The difference between two logarithms is very nearly proportional to the difference between the corresponding numbers when this difference is small.

The error made by assuming that these differences are exactly proportional will be so small that it may be neglected

Ex. 1. Find log 27845.

Omitting, for the moment, the decimal points from the mantissas, we have

mantissa of log 27850 = 44483,

mantissa of log 27840 = 44467,

difference of mantissas = 16.

Let x stand for the difference between the mantissas of log 27845 and log 27840; that is, for the *correction* to be added to the smaller mantissa to give the required mantissa.

Then, by the above property,

$$\frac{x}{16}=\frac{27845-27840}{27850-27840}=\frac{5}{10}=.5.$$

Whence $$x=.5\times 16=8.$$

Therefore, mantissa of log $27845=44467+8=44475$,

and $$\log 27845=4.44475.$$

Observe that, by Art. 16, the mantissa of log 27850 is the same as the mantissa of log 2785. In subsequent work such ciphers will be omitted.

The method can now be stated more concisely for practical work:

Subtract the mantissa corresponding to the first four figures of the given number from the next mantissa in the table; multiply this difference by the remaining figure or figures of the given number, treated as a decimal; add the product to the first (and smaller) mantissa.

Prefix finally the proper characteristic.

In thus finding the mantissa, a decimal point in the given number is ignored, in accordance with Art. 16.

The difference between two consecutive mantissas in the table is called the **Tabular Difference.**

Ex. 2. Find log 78.1283.

We have mantissa of log $7813=89282$,

mantissa of log $7812=89276$,

tabular difference $=\quad 6$,

correction $=.83\times 6=4.98$,

mantissa of log $781283=89276+5=89281$.

Therefore $$\log 78.1283=1.89281$$

Observe that the correction added to the mantissa of log 7812 is 5, the nearest integer to 4.98.

22. In the table of logarithms a column containing the required corrections (head **Pp. Pts.**, *i.e.*, proportional parts) is given. In this column there are several small tables, each containing two columns of numbers. One of these columns consists of the consecutive numbers 1 to 9; the other, headed by a tabular difference, contains the correction corresponding to each one of the figures 1 to 9, when it is the *fifth* figure of the number whose logarithm is required. When it is the *sixth* figure, the corresponding tabular correction must evidently be divided by 10; when it is the *seventh* figure, by 100; and so on.

Thus, in Ex. 1 of the preceding article, we take the correction opposite 5, under the tabular difference 16, and obtain 8, as before.

In Ex. 2, we take the following corrections from the column headed by the tabular difference 6:

$$\begin{array}{lr} \text{for } 8, & \text{correction} = 4.8 \\ \text{for } 3, & \text{correction} = \underline{0.18} \\ & \text{final correction} = 4.98, \text{ as before.} \end{array}$$

Observe that the correction for the sixth figure of the given number does not affect the result.

Ex. 3. Find the log .0128546.

$$\begin{array}{lr} \text{We have} & \text{mantissa of } \log 1286 = 10924, \\ & \text{mantissa of } \log 1285 = 10890, \\ & \text{tabular difference} = \quad 34. \end{array}$$

From the column of proportional parts headed by 34, we obtain:

$$\begin{array}{r} \text{correction for fifth figure } 4 = 13.6 \\ \text{correction for sixth figure } 6 = \underline{\ 2.04} \\ \text{total correction} = 15.64 \end{array}$$

Therefore, mantissa of log 128546 = 10890 + 16 = 10906,

and $\log .0128546 = 8.10906 - 10.$

Observe that in this example the correction for the sixth figure does affect the result.

EXERCISES III.

Verify the following statements:

1. $\log 13 = 1.11394$.
2. $\log 14.84 = 1.17143$.
3. $\log 73000 = 4.86332$.
4. $\log 5884.4 = 3.76970$
5. $\log .031586 = 8.49949 - 10$.
6. $\log .00391857 = 7.59313 - 10$.

Find the logarithms of each of the following numbers:

7. 5.
8. 18.
9. 540.
10. 3876.
11. 2076.
12. 59.80.
13. 1.87.
14. .01832.
15. .0004129.
16. 63072.
17. 59.836.
18. 4376.4.
19. .070518.
20. 185462.
21. .00103987.

To find a Number from its Logarithm.

23. Mantissa given in the Table. — If the mantissa of the given logarithm is found in the table, the first three figures of the required number will be in the same line with it in the column headed N, and the fourth figure over the column in which the given mantissa stands.

The characteristic is determined by Art. 15 (iii.) and (iv.).

Ex. 1. Find the number whose logarithm is 4.82099. The mantissa .82099 corresponds to the number 6622; but since the given characteristic is 4, the required number must have five integral places.

Consequently $\qquad 4.82099 = \log 66220.$

Ex. 2. Find the number whose logarithm is $8.78625 - 10$. The mantissa .78625 corresponds to the number 6113; but since the characteristic is -2, the required number must be a decimal having its first significant figure in the second decimal place.

Consequently $\qquad 8.78625 - 10 = \log .06113.$

24. Mantissa not given in the Table. — The method employed is the converse of that used in Art. 21 to find the logarithms of numbers that consist of more than four significant figures.

Ex. 1. Find the number whose logarithm is 2.81727.

We have

$$\begin{aligned} \text{given mantissa} &= 81727; \\ \text{next smaller mantissa} &= 81723, \text{ corresponding number} = 6565; \\ \text{next larger mantissa} &= 81730, \text{ corresponding number} = 6566. \end{aligned}$$

Let x stand for the difference between 6565 and the required number; that is, for the correction to be added to 6565.

We then have

$$\frac{x}{6566-6565}=\frac{81727-81723}{81730-81723}, \text{ or } \frac{x}{1}=\frac{4}{7}=.6,$$

corrected for the first decimal place. Notice that the significance of the decimal point in the result is that the correction is to be *annexed as an additional figure* to the smaller number.

Therefore, the figures in the required number are 65656; and since the characteristic of the given logarithm is 2, there are only three integral places. Hence $2.81727 = \log 656.56$.

This process may also be stated concisely for practical work:

Take the mantissa next smaller and the mantissa next larger than the given mantissa, and note the numbers corresponding; next divide the difference between the given mantissa and the next smaller by the difference between the next larger and the next smaller. Annex the quotient to the number corresponding to the smaller mantissa, neglecting the decimal point of the quotient.

Place the decimal point in the number thus obtained as it is determined by the given characteristic.

Ex. 2. Find the number whose logarithm is $7.18281 - 10$.

We have

$$\begin{aligned} \text{given mantissa} &= 18281; \\ \text{next smaller mantissa} &= 18270, \text{ corresponding number} = 1523; \\ \text{next larger mantissa} &= 18298, \text{ corresponding number} = 1524. \end{aligned}$$

Hence the correction to be annexed to 1523 is

$$\frac{18281-18270}{18298-18270}, =\frac{11}{28}, =.39+$$

Therefore the figures of the required number are 152339; and since the characteristic of the given logarithm is -3, there must be two ciphers between the decimal point and the first significant figure.

Consequently $7.18281 - 10 = \log .00152339$.

In general, in using a five-place table, the numbers corresponding to given mantissas should be carried to only *five* significant figures, as in Ex. 1.

But with mantissas in the first two pages of the table, the corresponding numbers may be carried to six figures. The reason being that the tabular differences later become so small that the correction for a sixth figure will not in general affect the result. See Exx. 2–3, Art. 22.

25. The correction to be added to the number corresponding to the next smaller mantissa may also be taken from the column of proportional parts.

In this column turn to the table headed by the number which is equal to the difference between the next larger and the next smaller mantissa. As the first figure of the correction take the figure in this table which is opposite the proportional part nearest to the difference between the given mantissa and the next smaller mantissa.

If a second figure in the correction is to be found, we should take as the first figure that figure which is opposite the proportional part *next smaller* than the difference between the given mantissa and the next smaller.

Multiply by 10 the difference between the proportional part already used and the difference between the given mantissa and the next smaller, and take the product as a proportional part in determining the second figure of the correction; and so on.

Thus, in Ex. 1 of the preceding article, we turn to the column headed by the tabular difference 7. The proportional part in this table that is nearest to 4 (the difference between the given mantissa and the next smaller) is 4.2; the number opposite 4.2 is 6, the correction previously obtained.

In Ex. 2, we turn to the column headed by the tabular difference 28. The proportional part *next smaller* than 11 (the difference between the given mantissa and the next smaller) is 8.4; the figure opposite 8.4 is 3, the first figure of the correction.

We next multiply 2.6 $(= 11 - 8.4)$ by 10, and take the product 26 as a proportional part. The figure opposite 25.2 (nearest to 26) in the column headed by 28 is 9, the second figure of the correction. Therefore, the required correction is found to be 39, as before.

EXERCISES IV.

Verify the following statements:

1. $\log x = 3.14926,\quad x = 1410.13.$
2. $\log x = 1.59187,\quad x = 39.073.$
3. $\log x = .34159,\quad x = 2.1958.$
4. $\log x = 9.57187 - 10,\quad x = .37314.$
5. $\log x = 7.83957 - 10,\quad x = .0069115.$
6. $\log x = 6.18953 - 10,\quad x = .00015471.$

Find the numbers whose logarithms are:

7. 2.26150.	8. .59726.	9. 8.94655 − 10.
10. 3.88825.	11. 6.19815.	12. 6.72576 − 10.
13. 4.98880.	14. 1.68417.	15. 9.23360 − 10.

Cologarithms.

26. The **Cologarithm** of a number, or, as it is sometimes called, the *Arithmetical Complement* of the logarithm, is defined as the logarithm of the reciprocal of the number.

That is, $\operatorname{colog} n = \log \frac{1}{n} = \log 1 - \log n = 0 - \log n.$

We thus see that the cologarithm of a number is obtained by subtracting its logarithm from 0. But this step would leave the mantissa as well as the characteristic negative. To avoid a negative mantissa, therefore, we subtract the logarithm from $10 - 10, = 0$.

Ex. 1. Find the colog 3.

Subtracting log 3, $=.47712$, from $10-10$, we have

$$\begin{array}{r} 10. \qquad -10 \\ .47712 \qquad \\ \hline 9.52288-10 \end{array}$$

Therefore $\text{colog } 3 = 9.52288 - 10.$

Ex. 2. Find colog .0054.

Subtracting log .0054, $= 7.73239 - 10$, from $10 - 10$, we have

$$\begin{array}{r} 10. \qquad -10 \\ 7.73239-10 \\ \hline 2.26761 \qquad \end{array}$$

Therefore $\text{colog } .0054 = 2.26761.$

EXERCISES V.

Verify the following statements:

1. $\text{colog } 543 = 7.26520 - 10.$
2. $\text{colog } 72.318 = 8.14075 - 10.$
3. $\text{colog } 8.9134 = 9.04996 - 10.$
4. $\text{colog } .38145 = .41856.$
5. $\text{colog } .051984 = 1.28413.$
6. $\text{colog } .0091437 = 2.03887.$

Find the cologarithm of each of the following numbers:

7. 5817. 8. .6305. 9. .009812. 10. 763.85.

11. 15.482. 12. 7.00386. 13. .000594. 14. 32581.9

Applications.

27. Ex. 1. Compute the value of x, when

$$x = 53.847 \times .0085965.$$

$$\log x = \log 53.847 + \log .0085965.$$

$$\begin{aligned} \log 53.847 &= 1.73117 \\ \log .0085965 &= \underline{7.93433 - 10} \\ \log x &= 9.66550 - 10 \end{aligned}$$

$$x = .46291.$$

Ex. 2. Compute the value of x, when

$$x = 8.4394 \div .31416.$$

$$\log x = \log 8.4394 + \operatorname{colog} .31416.$$

$$\begin{aligned} \log 8.4394 &= .92631 \\ \operatorname{colog} .31416 &= \underline{.50285} \\ \log x &= 1.42916 \\ x &= 26.863. \end{aligned}$$

Ex. 3. Compute the value of x, when

$$x = \frac{6.4319 \times .59218}{7.9254 \times .062547}.$$

$$\log x = \log 6.4319 + \log .59218 + \operatorname{colog} 7.9254 + \operatorname{colog} .062547$$

$$\begin{aligned} \log 6.4319 &= .80834 \\ \log .59218 &= 9.77246 - 10 \\ \operatorname{colog} 7.9254 &= 9.10098 - 10 \\ \operatorname{colog} .062547 &= \underline{1.20379 } \\ \log x &= 20.88557 - 20 \\ &= .88557. \\ x &= 7.6837. \end{aligned}$$

Ex. 4. Find the value of x, when

$$x = .5318^4.$$

$$\begin{aligned} \log x &= 4 \log .5318 \\ &= 4(9.72575 - 10) \\ &= 38.90300 - 40 \\ &= 8.90300 - 10. \\ x &= .079983. \end{aligned}$$

Ex. 5. Find the value of $\sqrt[3]{-.031459}$.

Since a negative number cannot be expressed as a power of $+10$, such a number does not have a logarithm. In this example, therefore, and in all similar examples, we first determine the sign of the result. We then find the value of the expression obtained by changing each sign $-$ to $+$, and to that result prefix the sign previously determined.

The sign of the result of this example is $-$

Let $x = \sqrt[3]{.031459}.$

Then
$$\begin{aligned}\log x &= \tfrac{1}{3}\log .031459\\ &= \tfrac{1}{3}(28.49775 - 30)\\ &= 9.49925 - 10,\end{aligned}$$

and $x = .31568.$

Therefore, the required result is $-.31568$.

Observe that in dividing log .031459 by 3, we first modified the characteristic so that the number, 30, which is subtracted from the logarithm is 10 times the divisor; that is, so that the quotient obtained by dividing this number by 3 is 10.

Ex. 6. Compute the value of x, when

$$x = \frac{4.5921 \times \sqrt[3]{.021946}}{.059318 \times .41587^3}.$$

$$\log x = \log 4.5921 + \tfrac{1}{3}\log .021946 + \operatorname{colog} .059318 + 3 \operatorname{colog} .41587.$$

$$\begin{aligned}\log 4.5921 &= .66201\\ \tfrac{1}{3}\log .021946 = \tfrac{1}{3}(28.34135 - 30) &= 9.44712 - 10\\ \operatorname{colog} .059318 &= 1.22681\\ 3\operatorname{colog} .41587 = 3 \times .38104 &= \underline{1.14312}\\ \log x &= 12.47906 - 10\\ &= 2.47906.\\ x &= 301.34.\end{aligned}$$

Ex. 7. Compute the value of x, when

$$x = \sqrt[3]{\frac{5.4318 \times \sqrt{.31459}}{7.1938 \times .2934^2}}.$$

For convenience in arranging the logarithmic work, we first cube both members of this equation, and obtain

$$x^3 = \frac{5.4318 \times \sqrt{.31459}}{7.1938 \times .2934^2}. \tag{1}$$

Taking logarithms, we have

$$3 \log x = \log 5.4318 + \tfrac{1}{2} \log .31459 + \operatorname{colog} 7.1938 + 2 \operatorname{colog} .2934. \quad (2)$$

In practice, step (1) should be performed mentally, and the result (2) be at once written.

$$\begin{aligned}
\log 5.4318 &= .73494 \\
\tfrac{1}{2} \log .31459 = \tfrac{1}{2}(19.49775 - 20) &= 9.74887 - 10 \\
\operatorname{colog} 7.1938 &= 9.14304 - 10 \\
2 \operatorname{colog} .2934 = 2 \times 0.53254 &= \underline{1.06508} \\
3 \log x &= 20.69193 - 20 \\
&= .69193. \\
\log x &= .23064. \\
x &= 1.70076.
\end{aligned}$$

EXERCISES VI.

Find the values of each of the following expressions:

1. 31.834×185.592.
2. $8.0043 \times .5319$.
3. $.004893 \times 6.5942$.
4. $(-.0514) \times .123857$.
5. $\dfrac{.78}{347}$.
6. $\dfrac{1539}{78395}$.
7. $\dfrac{19.7939}{3892.7}$.
8. $\dfrac{380.14 \times (-.0576)}{7.3792}$.
9. $\dfrac{(-9.7408) \times .000395}{36.937}$.
10. $\dfrac{5.83 \times 91.358}{.00479}$.
11. $\dfrac{57.13 \times 9.0047}{5.382 \times .07235}$.
12. $\dfrac{4.9 \times (-306) \times 48.3}{100.088 \times 2.9 \times .081}$.
13. $\dfrac{.79 \times 891.3 \times .00099}{(-10.236) \times .07 \times .0031}$.
14. 7.0435^3.
15. $.31844^4$.
16. 2.3817^5.
17. $(3.68 \times .97)^4$.
18. $(.7918 \times 3.17)^5$.
19. $[.034 \times (-4.9738)]^4$.
20. $(17.19 \times .00001986)^4$.

21. $\sqrt[5]{13}$. **22.** $\sqrt[7]{-251}$. **23.** $\sqrt[8]{39.837}$.

24. $\sqrt[10]{163.4^3}$. **25.** $\sqrt[5]{.31492^2}$. **26.** $\sqrt[6]{1.0031}$.

27. $\sqrt[3]{\frac{14}{15}}$. **28.** $\sqrt[4]{\frac{11}{17}}$. **29.** $\sqrt[5]{\frac{21}{814}}$.

30. $\frac{2}{3}\sqrt[3]{\frac{5}{6}}$. **31.** $2\frac{1}{2}\sqrt[4]{\frac{7}{8}}$. **32.** $7\frac{3}{4}\sqrt[6]{\frac{2}{3}}$.

33. $(.74\sqrt[3]{8.21})^4$. **34.** $(5.21\sqrt[5]{.3817})^6$.

35. $\frac{3}{4}\sqrt[3]{-5} \times \sqrt[4]{17}$. **36.** $3\frac{5}{6}\sqrt[4]{.38} \times \sqrt[5]{7.3815}$.

37. $\sqrt[3]{(.25\sqrt{3})}$. **38.** $\sqrt[5]{(112.34\sqrt[3]{.003914})}$.

39. $\sqrt[8]{(17.2\sqrt[3]{.718})}$. **40.** $\sqrt[11]{(-23\sqrt[7]{.18943})}$.

41. $5.341\sqrt[4]{(27.39\sqrt[3]{.1439})}$. **42.** $23.491^2\sqrt[5]{(.18\sqrt[4]{17.3})}$.

43. $\sqrt[6]{\dfrac{3.19\sqrt[5]{-9.2614}}{.519^2\sqrt{117.38}}}$. **44.** $5.14\sqrt[7]{\dfrac{.1934\sqrt[3]{.13945}}{583.5\sqrt{27.3}}}$.

Exponential Equations.

28. An Exponential Equation is an equation in which the unknown number appears as an exponent of a known or an unknown number, as $a^x = b$.

Ex. 1. Solve the equation $3^x = 9$.

Taking logarithms, $x \log 3 = \log 9 = 2 \log 3$.

Hence $x = 2$.

This result could have been obtained by inspection, by writing the given equation $3^x = 3^2$.

Ex. 2. Find the value of x in $3^x = 5$.

$$3^x = 5;$$

taking logarithms, $x \log 3 = \log 5$;

whence $$x = \frac{\log 5}{\log 3} = \frac{.69897}{.47712} = 1.46497.$$

Ex. 3. Find the value of x in the following equation

$$2^{3x+1} = 7^{2x-1};$$

taking logarithms, $(3x+1)\log 2 = (2x-1)\log 7$.

Removing parenthesis, $3x\log 2 + \log 2 = 2x\log 7 - \log 7$,

or $$x(3\log 2 - 2\log 7) = -\log 7 - \log 2;$$

whence
$$x = \frac{\log 7 + \log 2}{2\log 7 - 3\log 2}$$
$$= \frac{.84510 + .30103}{1.69020 - .90309}$$
$$= \frac{1.14613}{.78711} = 1.4561.$$

EXERCISES VII.

Solve the following exponential equations:

1. $2^x = 64$.
2. $3^x = 81$.
3. $2^{x-1} = .5^{3x-8}$.
4. $(-8)^{-x} = 16$.
5. $4^{3x-1} = .5^{x-5}$.
6. $4^x = 8$.
7. $8^x = 32$.
8. $5^x = (\sqrt{5})^{-1}$.
9. $4^{x+1} = 8 \cdot 2^{x+2}$
10. $25^{3x-1} = 625 \cdot 5^{x+8}$.
11. $7^{\sqrt{(x-3)}} = 343^{-1} \cdot 49^{\sqrt{(x-3)}}$.
12. $27^{\sqrt{(x-3)}} = (\sqrt{3})^{2\sqrt{(x+3)}}$.
13. $\sqrt{a^{11-x}} = a^{8-x}$.
14. $\sqrt[3]{a^{x+2}} = \sqrt{a^{x-3}}$.
15. $\sqrt{a^{3-4x}} \div \sqrt[5]{a^{6-7x}} \times a^{\frac{9}{2}} = 1$
16. $(\frac{1}{5})^x = 25$.
17. $(\frac{1}{2})^{x-7} = 64$.
18. $(\frac{27}{19})^{11x-5} = (\frac{19}{27})^{7x-8}$.
19. $(\frac{4}{3})^{4x-7} = .75^{2-3x}$.
20. $4^x - 6 \cdot 2^x + 8 = 0$.
21. $9^x + 243 = 36 \cdot 3^x$.
22. $3^{\log x} = 9$.
23. $5^{\log 2x} = 625$.
24. $16^{\log 3x} = 32^{\log x}$.
25. $5^x = 10$.
26. $16^x = 45$.
27. $11^x = 310$.
28. $25^x = 10$.
29. $7^x = 300$.
30. $3.594^x = 359600$.
31. $\sqrt[x]{9.8926} = 1.29$.
32. $5^x = 7^{3.14}$.
33. $x^{\sqrt{2}} = \sqrt[3]{3}$.
34. $5^{x+3} = 1000$.
35. $7^{x+1} = 5$.
36. $1.58^{x-5} = 9.847$.
37. $5^{x+1} = 11^{x-1}$.
38. $3^{x+7} = 7^{x+3}$.
39. $31^{x+3} = 25^{x+4}$.
40. $35^{x+2} = 40^{x-1}$.

Logarithmic Equations.

29. Ex. 1. Solve the equation $\frac{1}{2}\log(x-9)+\log\sqrt{(2x-1)}=1$

By the principles of logarithms, we obtain successively

$$\log\sqrt{(x-9)}+\log\sqrt{(2x-1)}=\log 10,$$
$$\log\sqrt{[(x-9)(2x-1)]}=\log 10.$$

Therefore $\sqrt{[(x-9)(2x-1)]}=10,$

or $2x^2-19x+9=100.$

The roots of this equation are 13 and $-\frac{7}{2}$.

Ex. 2. Solve the equation

$$\log(x+12)-\log x=0.8451+\log(6-5x).$$

By the principles of logarithms,

$$\log\frac{x+12}{x}=\log 7(6-5x), \text{ since } 0.8451=\log 7.$$

Consequently $\dfrac{x+12}{x}=42-35x$, or $x+12=42x-35x^2$.

The roots of this equation are $\frac{4}{7}$ and $\frac{21}{35}$.

Ex. 3. Solve the equation $x^{\log x}=100x$.

Taking logarithms, we obtain

$$(\log x)^2=\log 100+\log x, \text{ or } (\log x)^2-\log x=2.$$

Solving this equation as a quadratic in $\log x$, we obtain

$$\log x=2, \text{ or } x=100;\ \log x=-1, \text{ or } x=\tfrac{1}{10}.$$

EXERCISES VIII.

Solve the following logarithmic equations:

1. $\log x+\log(x+3)=1.$
2. $\log 4+2\log x=2.$
3. $\log 8+3\log x=3.$
4. $2\log x=1+\log(x+\frac{11}{10})$
5. $\log\sqrt{(7x+5)}+\log\sqrt{(2x+3)}=1+\log\frac{9}{2}.$
6. $\log(7-9x)^2+\log(3x-4)^2=2.$
7. $\log(x+\sqrt{x})+\log(x-\sqrt{x})=\log 4+\log x^2-\log x.$
8. $\dfrac{\log x^2}{\log(3x-16)}=2.$
9. $\dfrac{\log(2x-3)}{\log(4x^2-15)}=\frac{1}{2}.$
10. $\dfrac{\log(35-x^3)}{\log(5-x)}=3.$

Compound Interest and Annuities.

30. To find the compound interest, I, and the amount, A, of a given principal, P, for n years at r per cent.

If the interest is payable annually, the amount of $1 at the end of one year will be $1 + r$ dollars, and the amount of P dollars will be $P(1 + r)$ dollars. This amount, $P(1 + r)$, becomes the principal at the beginning the second year. Therefore, at the end of the second year the amount will be $P(1 + r) \times (1 + r), = P(1 + r)^2$ dollars, and so on.

Therefore, at the end of n years the amount will be $P(1 + r)^n$ dollars, or

$$A = P(1 + r)^n.$$

31. This formula can be used not only to find A, but also to find P, r, or n, when the three other quantities are given. Thus,

$$P = \frac{A}{(1 + r)^n}.$$

32. An **Annuity** is a fixed sum of money, payable yearly, or at other fixed intervals, as half-yearly, once in two years, etc.

33. To find the present value, P, of an annuity of A dollars, payable yearly for n years, at r per cent.

The present worth of the first payment is $\dfrac{A}{1 + r}$ dollars, of the second payment is $\dfrac{A}{(1 + r)^2}$ dollars, and, in general, of the nth payment is $\dfrac{A}{(1+r)^n}$ dollars.

Therefore the present worth of all the payments is

$$\frac{A}{1 + r} + \frac{A}{(1 + r)^2} + \cdots + \frac{A}{(1 + r)^n} = \frac{\dfrac{A}{1 + r}\left[1 - \left(\dfrac{1}{1 + r}\right)^n\right]}{1 - \dfrac{1}{1 + r}}.$$

Multiplying numerator and denominator by $1 + r$, we have

$$P = \frac{A}{r}\left[1 - \frac{1}{(1 + r)^n}\right].$$

Ex. 1. Find the amount of \$500 for 8 years at 5% compound interest.

$$A = P(1+r)^n = 500 \times 1.05^8.$$

$$\log A = \log 500 + 8 \log 1.05.$$

$$\begin{aligned} \log 500 &= 2.69897 \\ 8 \log 1.05 &= \underline{\ \ .16952} \\ \log A &= 2.86849 \\ A &= 738.73. \end{aligned}$$

Therefore the required amount is \$738.73.

Ex. 2. Find the present value of an annuity of \$1000 for 6 years, if the current rate of interest is 5%.

$$P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] = \frac{1000}{.05}\left[1 - \frac{1}{1.05^6}\right].$$

We will first compute 1.05^6,

$$\begin{aligned} \log (1.05)^6 &= 6. \log 1.05 \\ &= 6 \times .02119 \\ &= .12714. \\ (1.05)^6 &= 1.34012. \end{aligned}$$

We then have

$$P = \frac{1000}{.05}\left[1 - \frac{1}{1.34012}\right] = 20000 \times \frac{.34012}{1.34012},$$

$$\log P = \log 20000 + \log .34012 + \text{colog}\, 1.34012.$$

$$\begin{aligned} \log 20000 &= \ \ 4.30103 \\ \log .34012 &= \ \ 9.53163 - 10 \\ \text{colog}\, 1.34012 &= \underline{\ \ 9.87286 - 10} \\ \log P &= 23.70552 - 20 \\ &= \ \ 3.70552. \\ P &= 5076. \end{aligned}$$

Therefore the present value of the annuity is \$5076.

EXERCISES IX.

Find the amount at compound interest:

1. Of $3600 for 5 years at $4\frac{1}{2}\%$.
2. Of $1875.50 for 8 years at 5%.
3. Of $12,350 for 6 years at $3\frac{1}{2}\%$.
4. Of $21,580 for 7 years 4 months at 4%.

Find the principal that will amount to:

5. $7913 in 5 years at 5% compound interest.
6. $14,770 in 10 years at $4\frac{1}{2}\%$ compound interest.
7. $11,290 in 8 years at 4% compound interest.
8. $11,090 in 6 years 6 months at 3% compound interest.
9. In what time, at 4%, will $8010 amount to $11,400 at compound interest?
10. In what time, at $4\frac{1}{2}\%$, will $3530 amount to $6023.57, if the interest is compounded semi-annually?

Find the rate of compound interest:

11. If $1110 amounts to $1640 in 8 years.
12. If $3750 amounts to $6070 in 14 years.

Find the present value of an annuity:

13. Of $1000 for 10 years, if the current rate of interest is 4%.
14. Of $1250 for 8 years, if the current rate of interest is $4\frac{1}{2}\%$.
15. Of $2500 for 10 years, if the current rate of interest is 5%.
16. Of $3000 for 12 years, if the current rate of interest is 6%.

CHAPTER XXIX.

PROBABILITY.

1. In this chapter we shall consider the likelihood that an event, about whose happening there is uncertainty, will happen, or fail to happen.

Thus, if a coin be tossed once, it may fall heads up, but it is not certain to so fall. It may fall tails up. One way of falling is as likely to happen as the other. Now, $\frac{1}{2}$ of the whole number of ways in which a coin can fall is heads up. It seems natural, therefore, to take $\frac{1}{2}$ as the mathematical expression of the likelihood, or probability, that the coin will fall heads up. Then, $\frac{1}{2}$ is also the probability that the coin will fall tails up.

Again, let 4 white balls and 6 red balls be placed in a box, and one ball be drawn at random. If the balls cannot be distinguished by the sense of touch, one ball is as likely to be drawn as any other. Now, one ball can be drawn in 10 different cases, in 4 of which a white ball can be drawn. That is, the number of cases in which a white ball can be drawn is $\frac{4}{10}, = \frac{2}{5}$, of the whole number of cases. We therefore take $\frac{2}{5}$ as the mathematical expression of the probability of drawing at random a white ball. The probability of not drawing a white ball, which is the same as the probability of drawing a red ball, is evidently $\frac{3}{5}$.

If data relating to the number of times an event has happened in a large number of cases be collected, these data will indicate quite surely how often the same event will happen in the same number of cases under similar conditions.

Thus, from tables used by life insurance companies, we find that of 95,965 healthy persons of sixteen, 95,293 have lived to

be seventeen. We therefore take $\frac{95298}{95965}$ as the probability that a person of sixteen, in good health, will live to be seventeen.

2. The considerations of the preceding article naturally lead to the following definitions:

The **Favorable Cases** are those in which an event can happen, or has happened in an extended number of cases.

The **Unfavorable Cases** are those in which the event can fail to happen, or has failed to happen in an extended number of cases.

The **Probability** that an event will happen is the ratio of the number of favorable cases to the whole number of cases.

Evidently the probability that an event will not happen is the ratio of the number of unfavorable cases to the whole number of cases.

If a be the number of cases in which an event can happen, and b be the number of cases in which it can fail to happen, and each case be equally likely to happen, we have:

$\frac{a}{a+b}$ *is the probability that the event will happen;*

$\frac{b}{a+b}$ *is the probability that the event will not happen.*

The **Odds** in favor of an event is defined as the ratio of the number of favorable cases to the number of unfavorable cases.

That is, $\frac{a}{b}$ *are the odds in favor of the event;*

in like manner, $\frac{b}{a}$ *are the odds against the event.*

3. Since an event is certain to happen or fail to happen, the number of ways favorable to its *happening-or-failing* is $a+b$. Therefore, the probability of the event's happening-or-failing, that is, *certainty*, is

$$\frac{a+b}{a+b}=\frac{a}{a+b}+\frac{b}{a+b}=1.$$

4. If P be the probability that an event will happen, it follows from the preceding article that $1-P$ is the probability that the event will not happen.

Ex. What is the probability of throwing at least 4 in a single throw with two dice?

The number of cases favorable to throwing at least 4 is the number of cases in which 4, 5, 6, $\cdots$, 12 can be thrown.

The number of unfavorable cases is the number of cases in which 2 and 3 can be thrown.

The required probability can be obtained most readily by first finding the probability of the event's not happening.

The sum 2 can be thrown in one case, 1, 1. The sum 3 can be thrown in two cases, 1, 2 and 2, 1. The two dice can be thrown in $6 \times 6, = 36$, different cases, counting 4, 5 and 5, 4, say, as different throws.

Therefore, the probability of *not* throwing a sum at least 4 is $\frac{3}{36}, = \frac{1}{12}$; and hence, the required probability is $1 - \frac{1}{12}, = \frac{11}{12}$.

5. Ex. A father of thirty-five has a son of twelve. What is the probability that both will be alive thirty years hence?

From the table of mortality given below, we find that of 82,581 persons of thirty-five, 46,754 live to be sixty-five; that of 98,650 persons of twelve, 77,012 live to be forty-two. Now, each of the 46,754 cases favorable to the father can be taken with each of the 77,012 cases favorable to the son. That is, the number of cases favorable to both is $46{,}754 \times 77{,}012$. For a similar reason, the whole number of cases is $82{,}581 \times 98{,}650$. Therefore, the required probability is $\frac{46{,}754 \times 77{,}012}{82{,}581 \times 98{,}650}$.

The value of this fraction to five decimals places is readily obtained by logarithms, and is .44198.

Mortality Table.

The following table is taken from the *Actuaries' Table of Mortality*, prepared from data furnished by seventeen English Life Insurance Offices. It is based on the record of 62,537 assurances, and has been generally adopted by American Companies.

Age.	Number Living.	Number Dying.	Age.	Number Living.	Number Dying.	Age.	Number Living.	Number Dying.
10	100,000	676	40	78,653	815	70	35,837	2,327
11	99,324	674	41	77,838	826	71	33,510	2,351
12	98,650	672	42	77,012	839	72	31,159	2,362
13	97,978	671	43	76,173	857	73	28,797	2,358
14	97,307	671	44	75,316	881	74	26,439	2,339
15	96,636	671	45	74,435	909	75	24,100	2,303
16	95,965	672	46	73,526	944	76	21,797	2,249
17	95,293	673	47	72,582	981	77	19,548	2,179
18	94,620	675	48	71,601	1,021	78	17,369	2,092
19	93,945	677	49	70,580	1,063	79	15,277	1,987
20	93,268	680	50	69,517	1,108	80	13,290	1,866
21	92,588	683	51	68,409	1,156	81	11,424	1,730
22	91,905	686	52	67,253	1,207	82	9,694	1,582
23	91,219	690	53	66,046	1,261	83	8,112	1,427
24	90,529	694	54	64,785	1,316	84	6,685	1,268
25	89,835	698	55	63,469	1,375	85	5,417	1,111
26	89,137	703	56	62,094	1,436	86	4,306	958
27	88,434	708	57	60,658	1,497	87	3,348	811
28	87,726	714	58	59,161	1,561	88	2,537	673
29	87,012	720	59	57,600	1,627	89	1,864	545
30	86,292	727	60	55,973	1,698	90	1,319	427
31	85,565	734	61	54,275	1,770	91	892	322
32	84,831	742	62	52,505	1,844	92	570	231
33	84,089	750	63	50,661	1,917	93	339	155
34	83,339	758	64	48,744	1,990	94	184	95
35	82,581	767	65	46,754	2,061	95	89	52
36	81,814	776	66	44,693	2,128	96	37	24
37	81,038	785	67	42,565	2,191	97	13	9
38	80,253	795	68	40,374	2,246	98	4	3
39	79,458	805	69	38,128	2,291	99	1	1

EXERCISES.

1. With one die, what is the probability of throwing 6? Not 6? 6 three times in succession?

2. In a single throw with two dice, what is the probability of throwing an even number? At least 8? Not more than 5?

3. The letters *a, e, f, r,* are placed at random in a line. What is the probability that *fear* or *fare* will be written? That both vowels will come together?

4. If 52 cards be dealt to four players, what is the probability that a particular player will receive the four aces?

5. From a box containing 4 red balls, 6 black balls, and 7 white balls, 3 balls are drawn at random. What is the probability of drawing one ball of each color? 2 black and 1 white? 3 red?

6. If 6 coins be tossed, what is the probability that they will fall 4 heads and 2 tails? 3 heads and 3 tails?

7. Nine persons are seated at random at a round table. What is the probability that A and B will be seated together? That C will be seated between A and B?

8. If 4 different volumes of history, 3 of mathematics, and 6 of literature be placed at random on a shelf, what is the probability that all the volumes in the same subject will be placed together?

9. From a box containing tickets numbered 1, 2, 3, ⋯, 20, three tickets are drawn at random. What is the probability of drawing 2, 3, 5? 2, 3, and not 5? Neither 2, 3, nor 5? All even numbers? Consecutive numbers?

10–18. What are the odds in favor of the events whose probabilities are required in Exx. 1–9?

Referring to the accompanying table of mortality, find the probabilities of the events in Exx. 19–21:

19. That a man of 45 will live to be 50. To be 60. To be 70. To be 80. That he will die within 5 years. Within 10 years. Within 20 years.

20. That a man of 90 will live one year. Two years. Three years. Four years. Five years. At least five years.

21. At marriage, a man and his wife are 25 and 21, respectively. What is the probability that they will live to celebrate their silver wedding? Their golden wedding?

22. A representative of a firm sailed, first cabin, on a steamer which had a crew of 150 men, and which carried 150 first cabin and 250 second cabin passengers. On the voyage a man was lost. What is the probability, *to the firm*, that he was their representative? What, when a later report states that he was a passenger? What, when a still later report states that he was a first cabin passenger?

CHAPTER XXX.

CONTINUED FRACTIONS.

1. If the numerator and denominator of $\frac{30}{43}$ be divided by the numerator, we have

$$\frac{30}{43}=\frac{30 \div 30}{43 \div 30}=\frac{1}{1+\frac{13}{30}}.$$

Reducing $\frac{13}{30}$, and subsequent fractions, in a similar way, we obtain

$$\frac{30}{43}=\cfrac{1}{1+\cfrac{13 \div 13}{30 \div 13}}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{4}{13}}}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}}$$

The complex fraction thus obtained is usually written more compactly thus:

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}.$$

Observe that in the last form the signs + are written on a line with the denominators to distinguish the complex fraction from the sum of common fractions. It is important to keep in mind that in both forms the numerator at any stage is the numerator of a fraction whose denominator is the entire complex fraction which is written below and to the right of that particular numerator.

2. **A Continued Fraction** is a fraction whose numerator is an integer, and whose denominator is an integer plus a fraction

whose numerator is an integer, and whose denominator is an integer plus a fraction, etc.

A continued fraction frequently occurs in connection with an integral term.

E.g.,
$$5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{+3\frac{1}{4}}}}=5+\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4}.$$

In such cases it is customary to call the entire mixed number the continued fraction.

The general form of a continued fraction, therefore, is:

$$n+\cfrac{n_1}{d_1+\cfrac{n_2}{d_2+\cfrac{n_3}{d_3+\cdots}}}=n+\frac{n_1}{d_1+}\frac{n_2}{d_2+}\frac{n_3}{d_3+}\cdots.$$

3. We shall confine ourselves in this chapter to continued fractions in which the numerators are all 1, and the denominators all positive integers; of the general form, therefore,

$$n+\frac{1}{d_1+}\frac{1}{d_2+}\frac{1}{d_3+}\cdots,$$

in which the d's are all positive integers, and n is a positive integer or 0.

The n and the d's are called **Partial Quotients.**

4. A **Terminating,** or **Finite Continued Fraction,** is one in which the number of partial quotients is limited, as in the example given above.

A **Non-terminating,** or **Infinite Continued Fraction,** is one in which the number of partial quotients is unlimited or infinite.

To Convert a Common Fraction into a Terminating Continued Fraction.

5. Compare the work in Art. 1 of reducing $\frac{30}{43}$ to a continued fraction, with the work of finding the G. C. M. of 30 and 43:

$$\begin{array}{l}
30)43(1 \\
\quad\;\; \underline{30} \\
\quad\;\; 13)30(2 \\
\qquad\quad\;\; \underline{26} \\
\qquad\qquad 4)13(3 \\
\qquad\qquad\;\;\; \underline{12} \\
\qquad\qquad\quad\; 1)4(4 \\
\qquad\qquad\qquad\; \underline{4}
\end{array}$$

Observe that the successive *quotients* in the latter process are the partial quotients of the continued fraction. This is as it should be, since a comparison of the two processes shows that the successive steps of division in getting the partial quotients are identical with those in finding the G. C. M.

The method is evidently perfectly general and may be applied to any common fraction. If the fraction be improper, the first quotient will be the integral part of the continued fraction, and the remaining quotients the successive partial quotients of the continued fraction proper.

Ex. Reduce $\frac{151}{45}$ to a continued fraction.

By the method of G. C. M., we have

$$\begin{array}{l}
45)151(3 \\
\quad\;\; \underline{135} \\
\qquad\; 16)45(2 \\
\qquad\qquad \underline{32} \\
\qquad\qquad 13)16(1 \\
\qquad\qquad\quad\;\; \underline{13} \\
\qquad\qquad\qquad 3)13(4 \\
\qquad\qquad\qquad\;\; \underline{12} \\
\qquad\qquad\qquad\quad 1)3(3 \\
\qquad\qquad\qquad\qquad \underline{3}
\end{array}$$

Therefore, $$\frac{151}{45} = 3 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3}.$$

To Reduce a Terminating Continued Fraction to a Common Fraction.

6. We have only to retrace the steps taken in the preceding article in forming a continued fraction from a common fraction. Thus,

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{4}{13}=\frac{1}{1+}\,\frac{13}{30}=\frac{30}{43}.$$

Evidently this method is also perfectly general.

We therefore conclude that any common fraction can be converted into a terminating continued fraction, and, conversely, that any terminating continued fraction can be reduced to a common fraction.

The latter reduction becomes laborious in the case of a continued fraction with many partial quotients, and a simpler method will now be given.

7. A Convergent of a continued fraction is that part of it obtained by stopping with a definite partial quotient.

Thus, in $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4},$$

the first convergent is $\frac{1}{1}$; the second is $\frac{1}{1+}\,\frac{1}{2},\ =\frac{2}{3}$;

the third is $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+},\ =\frac{7}{10};$$

the fourth is $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4},\ =\frac{30}{43}.$$

For convenience, we will call the integral term, when there is one, the *zeroth* convergent, so that the nth convergent will always end with the nth partial quotient. We will denote the successive convergents by $\frac{N_0}{D_0}, \frac{N_1}{D_1}, \frac{N_2}{D_2}$, etc.

The above convergents may then be written thus:

$$\frac{N_1}{D_1}=\frac{1}{1};\ \frac{N_2}{D_2}=\frac{2}{3};$$

$$\frac{N_3}{D_3}=\frac{7}{10}=\frac{3\times 2+1}{3\times 3+1}=\frac{3\times N_2+N_1}{3\times D_2+D_1};$$

$$\frac{N_4}{D_4}=\frac{30}{43}=\frac{4\times\ 7+2}{4\times 10+3}=\frac{4\times N_3+N_2}{4\times D_3+D_2}.$$

That is, to form the numerator of the third convergent, multiply the numerator of the second by the third partial quotient, and to the product add the numerator of the first convergent. To form the numerator of the fourth convergent, multiply the numerator of the third convergent by the fourth partial quotient, and to the product add the numerator of the second convergent. In like manner, form the denominators from the denominators of the two preceding convergents.

In general,

The numerator of any convergent after the second (after the first if there be a zeroth convergent) is formed by multiplying the numerator of the immediately preceding convergent by that partial quotient with which the convergent to be computed ends, and to the product adding the numerator of the second preceding convergent; the denominator of the same convergent is formed in like manner from the denominators of the two convergents immediately preceding.

The principle holds for the second convergent when there is a zeroth convergent; and in all cases for the third convergent.

For $$\frac{N_0}{D_0}=\frac{n}{1},\ \frac{N_1}{D_1}=n+\frac{1}{d_1}=\frac{nd_1+1}{d_1},$$

$$\frac{N_2}{D_2}=n+\frac{1}{d_1+}\,\frac{1}{d_2}=n+\frac{d_2}{d_1d_2+1}=\frac{nd_1d_2+n+d_2}{d_1d_2+1}$$

$$=\frac{d_2(nd_1+1)+n}{d_1d_2+1}=\frac{d_2N_1+N_0}{d_2D_1+D_0},$$

$$\frac{N_3}{D_3}=n+\frac{1}{d_1+}\,\frac{1}{d_2+}\,\frac{1}{d_3+}=n+\frac{1}{d_1+}\,\frac{d_3}{d_2d_3+1}=n+\frac{d_2d_3+1}{d_1d_2d_3+d_1+d_3}$$

$$=\frac{nd_1d_2d_3+nd_1+nd_3+d_2d_3+1}{d_1d_2d_3+d_1+d_3}$$

$$=\frac{d_3(nd_1d_2+n+d_2)+(nd_1+1)}{d_3(d_1d_2+1)+d_1}=\frac{d_3N_2+N_1}{d_3D_2+D_1}.$$

If the principle hold up to and including any convergent, it holds for the next convergent.

Suppose it holds up to and including the kth convergent. We then have

$$\frac{N_k}{D_k}=d+\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots+\frac{1}{d_k}=\frac{d_kN_{k-1}+N_{k-2}}{d_kD_{k-1}+D_{k-2}}.$$

Now $$\frac{N_{k+1}}{D_{k+1}}=d+\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots+\cfrac{1}{d_k+\cfrac{1}{d_{k+1}}}$$

differs from the preceding convergent only in having $d_k+\dfrac{1}{d_{k+1}}$ as a denominator where the preceding has d_k. Making this substitution in $\dfrac{d_kN_{k-1}+N_{k-2}}{d_kD_{k-1}+D_{k-2}}$, we obtain an expression for $\dfrac{N_{k+1}}{D_{k+1}}$, without assuming that the principle holds beyond the kth convergent. Consequently,

$$\frac{N_{k+1}}{D_{k+1}}=\frac{\left(d_k+\dfrac{1}{d_{k+1}}\right)N_{k-1}+N_{k-2}}{\left(d_k+\dfrac{1}{d_{k+1}}\right)D_{k-1}+D_{k-2}}=\frac{d_kd_{k+1}N_{k-1}+N_{k-1}+d_{k+1}N_{k-2}}{d_kd_{k+1}D_{k-1}+D_{k-1}+d_{k+1}D_{k-2}}$$

$$=\frac{d_{k+1}(d_kN_{k-1}+N_{k-2})+N_{k-1}}{d_{k+1}(d_kD_{k-1}+D_{k-2})+D_{k-1}}=\frac{d_{k+1}N_k+N_{k-1}}{d_{k+1}D_k+D_{k-1}},$$

a result in accordance with the principle.

Therefore, since the principle holds to and including the third convergent, it holds for the fourth; then, since it holds for the fourth, it holds for the fifth; and so on.

This method of proof is called **Proof by Mathematical Induction**

Properties of Convergents.

8. (i.) *The successive convergents, beginning with the zeroth, are alternately less and greater than the continued fraction.*

Thus, from $$\frac{151}{45}=3+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3},$$

we have $$\frac{N_0}{D_0}=\frac{3}{1},\quad \frac{N_1}{D_1}=\frac{7}{2},\quad \frac{N_2}{D_2}=\frac{10}{3},\quad \frac{N_3}{D_3}=\frac{47}{14},\quad \frac{N_4}{D_4}=\frac{151}{45},$$

and $$\frac{3}{1}<\frac{151}{45},\quad \frac{7}{2}>\frac{151}{45},\quad \frac{10}{3}<\frac{151}{45},\quad \frac{47}{14}>\frac{151}{45}.$$

The symbol $\sim$, read *difference between,* is placed between two numbers to indicate that the less is to be subtracted from the greater. *E.g.*, $3\sim 4=4\sim 3=4-3=1$.

(ii.) *The difference between any two consecutive convergents is* 1 *divided by the product of their denominators.*

Thus,

$$\frac{3}{1}\sim\frac{7}{2}=\frac{1}{1\times 2},\quad \frac{7}{2}\sim\frac{10}{3}=\frac{1}{2\times 3},\quad \frac{10}{3}\sim\frac{47}{14}=\frac{1}{3\times 14},\ \text{etc.}$$

(iii.) *Each convergent is nearer in value to the continued fraction than any preceding convergent.*

Thus,

$$\frac{151}{45}\sim\frac{3}{1}=\frac{16}{45};\quad \frac{151}{45}\sim\frac{7}{2}=\frac{13}{90};\quad \frac{151}{45}\sim\frac{10}{3}=\frac{3}{135};$$

and $$\frac{16}{45}>\frac{13}{90}>\frac{3}{135}.$$

(iv.) *The convergents of even order continually increase, but are always less than the continued fraction; while the convergents of odd order continually decrease, but are always greater than the continued fraction.*

E.g., $$\frac{3}{1}<\frac{10}{3}<\frac{151}{45};\quad \frac{7}{2}>\frac{47}{14}.$$

In a terminating continued fraction, the last convergent will, of course, be the continued fraction, and therefore neither greater nor less than itself.

The proofs follow:

Let $$V = n + \frac{1}{d_1+}\,\frac{1}{d_2+}\,\frac{1}{d_3+}\cdots.$$

(i.) The zeroth convergent is too small by $\frac{1}{d_1+}\,\frac{1}{d_2+}\cdots$.

In the first convergent, the partial quotient d_1 is too small by $\frac{1}{d_2+}\cdots$; hence $\frac{1}{d_1}$ is too great, and therefore $n+\frac{1}{d_1}$ is too great.

In the second convergent, the second partial quotient d_2 is too small by $\frac{1}{d_3+}$; hence $\frac{1}{d_2}$ is too great, and therefore $d_1+\frac{1}{d_2}$ is also too great; finally $\frac{1}{d_1+}\,\frac{1}{d_2}$ is too small, and $n+\frac{1}{d_1+}\,\frac{1}{d_2}$ is too small. And so on.

(ii.) Since $$\frac{N_k}{D_k} \sim \frac{N_{k+1}}{D_{k+1}} = \frac{N_k D_{k+1} \sim D_k N_{k+1}}{D_k D_{k+1}},$$
we have only to prove
$$N_k D_{k+1} \sim D_k N_{k+1} = 1.$$

The law holds for the first two convergents.

For $$\frac{N_0}{D_0} \sim \frac{N_1}{D_1} = \frac{n}{1} \sim \frac{nd_1+1}{d_1} = \frac{nd_1 \sim (nd_1+1)}{1\cdot d_1} = \frac{1}{d_1}.$$

If it holds for any two consecutive convergents, it holds for the second of these two and the next convergent.

We have
$$N_k D_{k+1} \sim D_k N_{k+1} = N_k(d_{k+1}D_k + D_{k-1}) \sim D_k(N_k d_{k+1} + N_{k-1})$$
$$= N_k D_{k-1} \sim D_k N_{k-1}.$$

Therefore, if the principle holds for $\frac{N_{k-1}}{D_{k-1}} \sim \frac{N_k}{D_k}$,

it holds for $$\frac{N_k}{D_k} \sim \frac{N_{k+1}}{D_{k+1}}.$$

(iii.) $\frac{N_{k+1}}{D_{k+1}}$ differs from V only in having d_{k+1} where V has

$$d_{k+1} + \frac{1}{d_{k+2}} + \cdots, \ = K, \text{ say.}$$

Then, $$\frac{N_k}{D_k} = \frac{d_k N_{k-1} + N_{k-2}}{d_k D_{k-1} + D_{k-2}}, \quad V = \frac{KN_k + N_{k-1}}{KD_k + D_{k-1}}.$$

But $$\frac{N_k}{D_k} \sim V = \frac{N_k}{D_k} \sim \frac{KN_k + N_{k-1}}{KD_k + D_{k-1}} = \frac{N_k D_{k-1} \sim N_{k-1} D_k}{D_k(KD_k + D_{k-1})}$$

$$= \frac{1}{D_k(KD_k + D_{k-1})},$$

and $$\frac{N_{k-1}}{D_{k-1}} \sim V = \frac{N_{k-1}}{D_{k-1}} \sim \frac{KN_k + N_{k-1}}{KD_k + D_{k-1}} = \frac{K(N_{k-1}D_k \sim N_k D_{k-1})}{D_{k-1}(KD_k + D_{k-1})}$$

$$= \frac{K}{D_{k-1}(KD_k + D_{k-1})}.$$

But $K > 1$ and $D_{k-1} < D_k$. Therefore, $\frac{N_k}{D_k} \sim V < \frac{N_{k-1}}{D_{k-1}} \sim V.$

Hence, any convergent is nearer in value to the continued fraction than the immediately preceding convergent, and consequently than *any* preceding convergent.

(iv.) The proof follows at once from (iii.) and (i.).

Limit to Error of Any Convergent.

9. Since, by Art. 8 (i.), the value of a continued fraction is between the values of any two consecutive convergents, it must differ from either of them by less than they differ from each other.

Therefore, an error of taking $\frac{N_k}{D_k}$ for the continued fraction is, by Art. 8 (ii.), less than $\frac{1}{D_k D_{k+1}}$.

But $$D_{k+1} = d_{k+1} D_k + D_{k-1} > d_{k+1} D_k.$$

Therefore, the error of $\frac{N_k}{D_k}$ is less than $\frac{1}{d_{k+1} D_k^2}$.

Hence, to find a convergent which differs from the continued fraction by less than $\frac{1}{m}$, we have only to compute successive convergents up to $\frac{N_k}{D_k}$, wherein $d_{k+1}D_k^2 < m$.

Ex. Find an approximation to 3.14159, correct to five decimal places.

We have

$$3.14159 = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4}.$$

The successive convergents are $\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \cdots$.

The error of $\frac{355}{113}$ is less than $\frac{1}{25(113)^2}$ and with greater reason less than $\frac{1}{25(100)^2}$, $= .000004$.

Therefore $\frac{355}{113}$ is the required approximation.

To Reduce a Quadratic Surd to a Continued Fraction.

10. The general method may be illustrated by particular examples.

Ex. Reduce $\sqrt{14}$ to a continued fraction.

Since the greatest integer contained in $\sqrt{14}$ is 3, we assume

$$\sqrt{14} = 3 + \frac{1}{d_1}.$$

Then $d_1 = \dfrac{1}{\sqrt{14}-3} = \dfrac{\sqrt{14}+3}{14-9} = \dfrac{\sqrt{14}+3}{5}$.

Since the greatest integer in this value of d_1 is 1, we assume

$$d_1 = \frac{\sqrt{14}+3}{5} = 1 + \frac{1}{d_2}.$$

Then $d_2 = \dfrac{5}{\sqrt{14}-2} = \dfrac{5\,(\sqrt{14}+2)}{10} = \dfrac{\sqrt{14}+2}{2}$.

In like manner, we assume

$$d_2=\frac{\sqrt{14}+2}{2}=2+\frac{1}{d_3}.$$

Then $d_3=\dfrac{2}{\sqrt{14}-2}=\dfrac{2(\sqrt{14}+2)}{10}=\dfrac{\sqrt{14}+2}{5}=1+\dfrac{1}{d_4}.$

Similarly,

$$d_4=\frac{5}{\sqrt{14}-3}=\frac{5(\sqrt{14}+3)}{5}=\sqrt{14}+3=6+\frac{1}{d_5};$$

$$d_5=\frac{1}{\sqrt{14}-3},\text{ etc.}$$

Since this process may be continued indefinitely, we obtain an infinite continued fraction by substituting, in succession, the values obtained for $d_1, d_2, d_3, \cdots$.

We then have

$$\sqrt{14}=3+\frac{1}{d_1}=3+\frac{1}{1+}\frac{1}{d_2}=3+\frac{1}{1+}\frac{1}{2+}\frac{1}{d_3}$$

$$=3+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\cdots.$$

Observe that the value obtained for

$$d_5=\frac{1}{\sqrt{14}-3}$$

is the same as that for d_1, so that

$$d_6=d_2,\ d_7=d_3,\ d_8=d_4,\ d_9=d_5=d_1,\text{ etc.}$$

Therefore the partial quotients 1, 2, 1, 6, are repeated indefinitely.

11. A Periodic Continued Fraction is an infinite continued fraction in which the partial quotients are repeated in sets of one or more.

Ex. Reduce $\dfrac{5-\sqrt{3}}{6}$ to a periodic continued fraction.

Since $\dfrac{5-\sqrt{3}}{6}<1$, we assume $\dfrac{5-\sqrt{3}}{6}=\dfrac{1}{d_1}.$

Then $\qquad d_1=\dfrac{6}{5-\sqrt{3}}=\dfrac{3(5+\sqrt{3})}{11}=1+\dfrac{1}{d_2}.$

And $\quad d_2 = \dfrac{11}{4+3\sqrt{3}} = \dfrac{11(4-3\sqrt{3})}{-11} = \dfrac{3\sqrt{3}-4}{1} = 1 + \dfrac{1}{d_3}.$

Similarly, $d_3 = \dfrac{1}{3\sqrt{3}-5} = \dfrac{3\sqrt{3}+5}{2} = 5 + \dfrac{1}{d_4}.$

Likewise, $d_4 = \dfrac{2}{3\sqrt{3}-5} = 3\sqrt{3}+5 = 10 + \dfrac{1}{d_5}.$

Finally, $\quad d_5 = \dfrac{1}{3\sqrt{3}-5} = d_3.$

Therefore, only the third and fourth partial quotients are repeated, and the required fraction is

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{10+}\,\frac{1}{5+}\,\frac{1}{10+}\cdots.$$

Application of Convergents.

12. It is often convenient to substitute for a fraction with large terms, or for a quadratic surd, a convergent with comparatively small terms, provided that convergent approximates closely enough to the true value.

Ex. 1.
$$\tfrac{351}{1003} = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{50}}}}$$

By Art. 9, we should expect the third convergent to be a close approximation, since the following partial quotient, 50, is large.

We have $\quad \dfrac{N_1}{D_1} = \dfrac{1}{2},\ \dfrac{N_2}{D_2} = \dfrac{1}{3},\ \dfrac{N_3}{D_3} = \dfrac{7}{20}.$

Therefore, by Art. 9, the error of the third convergent is less than

$$\frac{1}{50 \times 20^2}, \ = .00005.$$

Consequently, $\frac{7}{20}$ represents the true value of $\frac{351}{1003}$ correctly to four decimal places.

Ex. 2. Given $\sqrt{14} = 3 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\cdots$, find the error of the seventh convergent.

The student may satisfy himself that $\frac{N_7}{D_7} = \frac{449}{120}$.

The error of $\frac{N_7}{D_7} < \frac{1}{6(120)^2} < \frac{1}{86400} < .000011\cdots$.

Therefore $\frac{449}{120}$ is correct to four decimal places.

To Reduce a Periodic Continued Fraction to an Irrational Number.

13. We will take as an example the result of Ex. Art. 10.

Assume $$x = 3 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\cdots,$$

then $$x - 3 = \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+}\cdots.$$

Since the partial quotients 1, 2, 1, 6 are repeated indefinitely in that order, the continued fraction whose first partial quotient is the first periodic number (*i.e.*, 1) at any stage, and which is continued indefinitely, differs in no respect from the given periodic continued fraction. For example, the periodic continued fraction which follows the heavy plus sign ($\boldsymbol{+}$), in the value of $x - 3$ above, is the same as the entire continued fraction, which is the value of $x - 3$.

We may therefore substitute $x - 3$ for the part of the continued fraction which follows that particular plus sign. We thus have

$$x - 3 = \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{6+x-3} = \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+x} = \frac{11+3x}{15+4x}.$$

From this equation we obtain

$$4x^2 = 56, \text{ or } x = \sqrt{14}.$$

14. If the continued fraction be not periodic from the beginning, we first reduce the periodic part by itself as above, and substitute its value in the given continued fraction. The latter is then a terminating continued fraction and can be reduced to a simple fraction, whose numerator and denominator will not, however, be rational.

Ex. $$x = \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{3+}\,\frac{1}{5+}\cdots,$$

the periodic part commencing with the third partial quotient

Assume $$y = \frac{1}{3+}\,\frac{1}{5+}\cdots = \frac{1}{3+}\,\frac{1}{5+y} = \frac{5+y}{16+3y};$$

hence $$3y^2 + 16y = 5 + y,$$

and $y = \dfrac{-15+\sqrt{285}}{6}$. But $x = \dfrac{1}{2+}\,\dfrac{1}{1+y} = \dfrac{77-\sqrt{285}}{166}$.

EXERCISES.

Compute the successive convergents to

1. $\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{3}$. **2.** $2+\dfrac{1}{5+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{4}$.

Reduce each of the following fractions to a continued fraction, find its convergents, and determine a limit to the error of the third convergent.

3. $\frac{25}{89}$. **4.** $\frac{14}{105}$. **5.** $\frac{85}{16}$. **6.** $\frac{258}{91}$. **7.** $\frac{819}{87}$.

8. $\frac{582}{1198}$. **9.** $\frac{2771}{578}$. **10.** $27\frac{48}{51}$. **11.** .4751. **12.** 5.0372.

Reduce each of the following surds to continued fractions, find the first five convergents, and determine a limit to the error of the fourth convergent.

13. $\sqrt{7}$. **14.** $\sqrt{23}$. **15.** $\sqrt{2.5}$. **16.** $\sqrt{29}$. **17.** $2\sqrt{45}$.

18. $\dfrac{1+\sqrt{2}}{5}$. **19.** $\dfrac{22-\sqrt{7}}{4}$. **20.** $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$. **21.** $\dfrac{11+\sqrt{7}}{5}$.

Reduce the following periodic continued fractions to surds:

22. $$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$$

23. $$3+\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\cdots.$$

24. Express the decimal 2.71828 as a continued fraction, find its seventh convergent, and determine a limit to the error of this convergent.

CHAPTER XXXI.

SUMMATION OF SERIES.

By Undetermined Coefficients.

1. When the nth term of a series is a rational, integral function of n, the sum of n terms can be found by means of undetermined coefficients. The form which the sum of n terms of an arithmetical progression assumes will suggest a method of procedure.

By Ch. XXI., Art. 10, the sum of n terms of the A. P.

$$3 + 5 + 7 + 9 + \cdots + [3 + (n - 1)2] \text{ is } 2n + n^2.$$

Now observe that the sum of n terms is an integral function of n, *of degree one higher than the nth term.*

In applying the method of undetermined coefficients, we start with this assumption.

Ex. Find the sum of n terms of the series

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) + \cdots$$

Since the nth term, $n(n + 1)$, is of the second degree, we assume

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = A + Bn + Cn^2 + Dn^3. \quad (1)$$

The validity of this assumption will be proved by mathematical induction. The method of proof will at the same time determine the values of A, B, C, D.

We have now to prove that if this relation hold for the sum of n terms, it holds for the sum of $n + 1$ terms. Evidently the latter sum will involve $n + 1$ just as the sum of n terms involves n. That is,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) + (n + 1)(n + 2)$$
$$= A + B(n + 1) + C(n + 1)^2 + D(n + 1)^3. \quad (2)$$

Subtracting (1) from (2), we have

$$(n+1)(n+2)=B(n+1)-Bn+C(n+1)^2-Cn^2+D(n+1)^3-Dn^3,$$

or $\quad 2+3n+n^2=(B+C+D)+(2C+3D)n+3Dn^2.$

This relation will hold, if

$$3D=1,\ \ 2C+3D=3,\ \ B+C+D=2;$$

that is, if $\qquad B=\frac{2}{3},\ \ C=1,\ \ D=\frac{1}{3}.$

The formula in the second number of (1) now stands

$$A+\tfrac{2}{3}n+n^2+\tfrac{1}{3}n^3.$$

This will hold for the first term, if

$$1\cdot 2=A+\tfrac{2}{3}+1+\tfrac{1}{3};\ \text{ that is, if } A=0.$$

Therefore,

$$1\cdot 2+2\cdot 3+\cdots+n(n+1)=\tfrac{2}{3}n+n^2+\tfrac{1}{3}n^3=\tfrac{1}{3}n(n+1)(n+2).$$

Since the formula $\frac{1}{3}n(n+1)(n+2)$ holds for the first term, it holds for the sum of two terms; then, since it holds for the sum of two terms, it holds for the sum of three terms; and so on, to the sum of any number of terms.

By separating Terms into Partial Fractions.

2. Ex. Find the sum of n terms of the series

$$\frac{1}{(x+1)(x+2)(x+3)}+\frac{1}{(x+2)(x+3)(x+4)}+\frac{1}{(x+3)(x+4)(x+5)}$$
$$+\cdots+\frac{1}{(x+n)(x+n+1)(x+n+2)}+\cdots.$$

We separate the nth term into its partial fractions and use the result as a formula.

We have, by Ch. XXVI., Art. 8, Ex. 5,

$$\frac{1}{(x+n)(x+n+1)(x+n+2)}=\frac{1}{2(x+n)}-\frac{1}{x+n+1}+\frac{1}{2(x+n+2)}.$$

Giving to n the values 1, 2, 3, $\cdots$, we obtain

$$\frac{1}{(x+1)(x+2)(x+3)}=\frac{1}{2\,(x+1)}-\frac{1}{x+2}+\frac{1}{2\,(x+3)},$$

$$\frac{1}{(x+2)(x+3)(x+4)}=\frac{1}{2\,(x+2)}-\frac{1}{x+3}+\frac{1}{2\,(x+4)},$$

$$\frac{1}{(x+3)(x+4)(x+5)}=\frac{1}{2\,(x+3)}-\frac{1}{x+4}+\frac{1}{2\,(x+5)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{(x+n)(x+n+1)(x+n+2)}=\frac{1}{2\,(x+n)}-\frac{1}{x+n+1}+\frac{1}{2\,(x+n+2)}.$$

We now have

$$S_2=\frac{1}{2\,(x+1)}-\frac{1}{2\,(x+2)}-\frac{1}{2\,(x+3)}+\frac{1}{2\,(x+4)},$$

$$S_3=\frac{1}{2\,(x+1)}-\frac{1}{2\,(x+2)}-\frac{1}{2\,(x+4)}+\frac{1}{2\,(x+5)},$$

and, in general,

$$\begin{aligned}S_n&=\frac{1}{2\,(x+1)}-\frac{1}{2\,(x+2)}-\frac{1}{2\,(x+n+1)}+\frac{1}{2\,(x+n+2)}\\&=\frac{1}{2}\left[\frac{1}{(x+1)(x+2)}-\frac{1}{(x+n+1)(x+n+2)}\right].\end{aligned}$$

As $$n \doteq \infty,\ S_n \doteq \frac{1}{2\,(x+1)(x+2)}.$$

Hence the given series is convergent.

It is sometimes possible to determine the partial fractions by inspection.

EXERCISES I.

Find the sum of n terms of each of the following series:

1. $1\cdot 3+3\cdot 5+5\cdot 7+\cdots$.
2. $1^2+2^2+3^2+\cdots$.
3. $1^3+3^3+5^3+\cdots$.
4. $1\cdot3\cdot5+3\cdot5\cdot7+5\cdot7\cdot9+\cdots$.
5. $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\cdots$.
6. $1\cdot 3^2+3\cdot 5^2+5\cdot 7^2+\cdots$.

Find the sum of n terms, and the limit of this sum as $n \doteq \infty$, of each of the following series:

7. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots.$

8. $\dfrac{1}{1\cdot 4}+\dfrac{1}{2\cdot 5}+\dfrac{1}{3\cdot 6}+\cdots.$

9. $\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{1}{3\cdot 5\cdot 7}+\dfrac{1}{5\cdot 7\cdot 9}+\cdots.$

10. $\dfrac{1}{1\cdot 3\cdot 4}+\dfrac{1}{2\cdot 4\cdot 5}+\dfrac{1}{3\cdot 5\cdot 6}+\cdots.$

11. $\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+2)(x+3)}+\dfrac{1}{(x+3)(x+4)}+\cdots.$

12. $\dfrac{1}{x(x+2)(x+3)}+\dfrac{1}{(x+1)(x+3)(x+4)}+\dfrac{1}{(x+2)(x+4)(x+5)}+\cdots.$

13. $\dfrac{1}{(1+x)(1+ax)}+\dfrac{a}{(1+ax)(1+a^2x)}+\dfrac{a^2}{(1+a^2x)(1+a^3x)}+\cdots.$

14. $\dfrac{2}{1\cdot 3}\cdot\dfrac{1}{3}+\dfrac{3}{3\cdot 5}\cdot\dfrac{1}{3^2}+\dfrac{4}{5\cdot 7}\cdot\dfrac{1}{3^3}+\cdots+\dfrac{n+1}{(2n-1)(2n+1)}\cdot\dfrac{1}{3^n}\cdots.$

15. $\dfrac{1}{1\cdot 3\cdot 5\cdot 7}+\dfrac{2}{3\cdot 5\cdot 7\cdot 9}+\cdots+\dfrac{n}{(2n-1)(2n+1)(2n+3)(2n+5)}$

Recurring Series.

3. In a geometrical progression any term after the first is formed by multiplying the preceding term by a constant multiplier, the *ratio* of the series. To form such a series it is sufficient to know the first term and the ratio.

Thus, given $a=2$, $r=3x$, the series is

$$2+6x+18x^2+54x^3+\cdots.$$

4. A geometrical progression is a particular instance of a more general class of series. To form such series the first two or more terms, and as many ratios, must be given, as explained in the following examples.

Ex. 1. Given the first two terms, $3+2x$, and the two ratios, x^2 and $-2x$; form the series.

To form the third term we multiply the first term by the

first ratio, the second term by the second ratio, and add the resulting products. We thus have

$$3 \cdot x^2 + 2x(-2x), = -x^2, \text{ the third term.}$$

To form the fourth term we multiply the second term by the first ratio, the third term by the second ratio, and add the resulting products. Then,

$$2x \cdot x^2 + (-x^2)(-2x), = 4x^3, \text{ the fourth term.}$$

And so on.

Therefore, the required series is

$$3 + 2x - x^2 + 4x^3 - 9x^4 + \cdots.$$

Ex. 2. Given the first three terms, $1 - 4x + 3x^2$, and the three ratios, $-2x^3$, x^2, $4x$; form the series.

To form the fourth term we multiply the first term by the first ratio, the second term by the second ratio, the third term by the third ratio, and add the resulting products.

Therefore,

$$1 \cdot (-2x^3) - 4x \cdot x^2 + 3x^2 \cdot 4x, = 6x^3, \text{ the fourth term.}$$

In like manner,

$$-4x(-2x^3) + 3x^2 \cdot x^2 + 6x^3 \cdot 4x, = 35x^4, \text{ the fifth term;}$$

and so on.

Therefore, the required series is

$$1 - 4x + 3x^2 + 6x^3 + 35x^4 + \cdots.$$

5. A Recurring Series is a series in which, from and after a definite term, each term is formed by multiplying each of two or more preceding terms by a constant multiplier, and adding the resulting products.

Such a series is also called a **Compound Geometrical Progression.**

A recurring series is said to be of the *first order*, if one ratio be used, as in an ordinary geometrical progression; of the *second order*, if two ratios be used; and so on.

6. If u_1, u_2, u_3 be any three consecutive terms of Ex. 1, Art. 4, then $u_3 = x^2u_1 - 2xu_2$, or $u_3 + 2xu_2 - x^2u_1 = 0$.

That is, any three consecutive terms of a recurring series of the second order are connected by a definite relation.

The expression $\qquad 1 + 2x - x^2,$

formed by taking the coefficients of u_3, u_2, u_1 in the above relation, is called the **Scale of Relation** of the series.

In like manner, it follows that any four consecutive terms of the series in Ex. 2, Art. 4, are connected by the relation

$$u_4 - 4xu_3 - x^2u_2 + 2x^3u_1 = 0.$$

Therefore, the scale of relation of the series is

$$1 - 4x - x^2 + 2x^3.$$

Observe that *the scale of relation is obtained by subtracting the ratios in reverse order from unity.*

7. The following examples will illustrate the method of finding the ratios, and the scale of relation of a recurring series, when a sufficient number of terms are given.

Ex. 1. Find the ratios and the scale of relation of the series,

$$2 + 5x - x^2 + 11x^3 - 13x^4 + 35x^5 - 61x^6 + \cdots.$$

It is evident that this is not a recurring series of the first order. If it be a series of the second order, then

$$2h + 5x \cdot k = -x^2, \tag{1}$$

$$5x \cdot h - x^2 \cdot k = 11x^3, \tag{2}$$

wherein h and k are the required ratios. Solving these equations for h and k, we obtain $h = 2x^2$, $k = -x$.

We find that these ratios give the fifth and following terms of the series. Therefore the series is of the second order, and the scale of relation is

$$1 - (-x) - 2x^2, = 1 + x - 2x^2.$$

Ex. 2. Find the ratios and the scale of relation of the series

$$1 - 3x - 2x^2 + 3x^3 + 10x^4 + 6x^5 - 17x^6 - \cdots.$$

This series is evidently not of the first order. Assuming tentatively that the series is of the second order, and proceeding as in Ex. 1, we find that the two ratios thus obtained do not give the fifth and following terms. We therefore try three ratios.

Then
$$h - 3x \cdot k - 2x^2 \cdot l = 3x^3,$$
$$-3x \cdot h - 2x^2 \cdot k + 3x^3 \cdot l = 10x^4,$$
$$-2x^2 \cdot h + 3x^3 \cdot k + 10x^4 \cdot l = 6x^5.$$

Whence,
$$h = -x^3,\ k = -2x^2,\ l = x.$$

These ratios give the seventh term. Therefore the series is of the third order, and the scale of relation is

$$1 - x + 2x^2 + x^3.$$

8. When the scale of relation involves two ratios, two equations are necessary in order to determine these ratios, as in Ex. 1, Art. 7. To form the first equation, three terms must be given, and to form the second equation, a fourth term must be given. Therefore, when the scale of relation involves two ratios, at least four terms of the series must be known.

In like manner, we infer that, when the scale of relation involves three ratios, at least six terms of the series must be given.

In general, when the scale of relation involves n ratios, at least $2n$ terms of the series must be given.

The Sum of a Recurring Series.

9. Let $\quad u_1 + u_2 + u_3 + \cdots + u_{n-2} + u_{n-1} + u_n + \cdots$
be a recurring series of the second order, and let h and k be the ratios.

Then,
$$\left.\begin{aligned} u_3 &= u_1 h + u_2 k, \\ u_4 &= u_2 h + u_3 k, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ u_n &= u_{n-2} h + u_{n-1} k. \end{aligned}\right\} \qquad (1)$$

Let S_n stand for the sum of n terms; that is,

$$S_n = u_1 + u_2 + u_3 + u_4 + \cdots + u_{n-2} + u_{n-1} + u_n. \qquad (2)$$

Now, substituting for each term, after the second, in (2) its value given in (1), we have

$$\begin{aligned} S_n &= u_1 + u_2 + (u_1h + u_2k) + (u_2h + u_3k) + \cdots \\ &\quad + (u_{n-2}h + u_{n-1}k), \\ &= u_1 + u_2 + h(u_1 + u_2 + \cdots + u_{n-2}) \\ &\quad + k(u_2 + u_3 + \cdots + u_{n-1}). \end{aligned}$$

The coefficient of h is evidently $S_n - u_{n-1} - u_n$;
and the coefficient of k is $S_n - u_1 - u_n$.

Therefore, $S_n = u_1 + u_2 + h(S_n - u_{n-1} - u_n) + k(S_n - u_1 - u_n)$,
and $(1 - h - k)S_n = u_1 + u_2 - ku_1 - (hu_{n-1} + hu_n + ku_n)$.

Whence, $S_n = \dfrac{u_1 + u_2 - ku_1}{1 - h - k} - \dfrac{hu_{n-1} + hu_n + ku_n}{1 - h - k}$.

10. If the given series be convergent, $u_n \doteq 0$, and $u_{n-1} \doteq 0$, as $n \doteq \infty$.

Therefore, $$\frac{hu_{n-1} + hu_n + ku_n}{1 - h - k} \doteq 0.$$

Conversely, if this fraction approach zero, as n increases indefinitely,

$$S_n \doteq \frac{u_1 + u_2 - ku_1}{1 - h - k}, \qquad (1)$$

a definite finite number, and the given series is convergent.

Consequently, the fraction in (1) may be taken as the sum of an infinite recurring series of the second order, when the series is convergent.

In like manner, for an infinite recurring series of the third order, we obtain

$$S_n = \frac{u_1 + u_2 + u_3 - ku_1 - l(u_1 + u_2)}{1 - h - k - l}$$
$$- \frac{h(u_{n-2} + u_{n-1} + u_n) + k(u_{n-1} + u_n) + lu_n}{1 - h - k - l}.$$

Then, as above, if the infinite recurring series be convergent, we have

$$S_n \doteq \frac{u_1 + u_2 + u_3 - ku_1 - l(u_1 + u_2)}{1 - h - k - l} \tag{2}$$

as n increases indefinitely.

11. Ex. 1. Find the sum of the infinite recurring series

$$2 + 5x - x^2 + 11x^3 - 13x^4 + \cdots.$$

From Ex., Art. 7, we have $h = 2x^2$, $k = -x$.
Therefore, by Art. 10, (1),

$$S_n \doteq \frac{2 + 5x - (-x)2}{1 - 2x^2 - (-x)} = \frac{2 + 7x}{1 + x - 2x^2},$$

for such values of x as make the given series convergent.

12. It is evident that the recurring series in the example of the preceding article can be obtained from its sum either by division or by the method of undetermined coefficients.

On this account the fraction obtained as the sum of an infinite recurring series is called the **Generating Fraction** or the **Generating Function** of the series.

Convergency of a Recurring Series.

13. The convergency or divergency of an infinite recurring series may be determined by the methods given in Ch. XXV., or by Art. 10.

It is therefore important to obtain the general term of such a series. When the denominator of the generating fraction can be resolved into real binomial factors, the general term can be found as in Ch. XXVI., Art. 9.

Ex. Examine the series $2 + 5x - x^2 + 11x^3 - 13x^4 + \cdots$.

The generating fraction of the series was found to be (Art. 11)

$$\frac{2 + 7x}{1 + x - 2x}.$$

The general term of the expansion of this fraction is, by Ch. XXVI., Art. 9, Ex. 1, $x^n[3+(-1)^{n+1}2^n]$.

The ratio of convergence is

$$\frac{x^n[3+(-1)^{n+1}2^n]}{x^{n-1}[3+(-1)^n 2^{n-1}]}, = x\cdot\frac{\dfrac{3}{2^n}+(-1)^{n+1}}{\dfrac{3}{2^n}+(-1)^n\cdot\dfrac{1}{2}}, \doteq -2x.$$

Therefore the given series is convergent when x is numerically less than $\frac{1}{2}$.

EXERCISES II.

Find the ratios and the sum of n terms of the following recurring series:

1. $1+2+5+12+29+\cdots$.
2. $1-3-5+1+11+\cdots$.
3. $3-7+15-31+63-127+255-\cdots$.
4. $2-3+5-21+60-188+577-\cdots$.

Find the ratios and the generating function of the following recurring series:

5. $2-x-3x^2+5x^3+x^4-\cdots$.
6. $1+3x-9x^3-9x^4+\cdots$.
7. $1-4x+6x^2-14x^3+26x^4-\cdots$.
8. $1+4x-5x^2+11\frac{1}{2}x^3-22\frac{1}{4}x^4+\cdots$.
9. $3-6x-9x^2-10x^3-10\frac{1}{3}x^4-\cdots$.
10. $5-2x+11x^2+16x^3+65x^4+178x^5+551x^6+\cdots$.
11. $1-2x+3x^2+10x^3+13x^4+22x^5+51x^6+\cdots$.
12. $2-3x-2x^2+16x^3-24x^4-16x^5+128x^6-\cdots$
13. $3+5x-6x^2-15x^3+8x^4+37x^5-10x^6-\cdots$.
14. $1-3x+7x^2+2x^3-25x^4+42x^5+14x^6-\cdots$.

15–19. Find the general term of each series in Exx. 7–11, and hence determine for what values of x each series is convergent.

Method of Finite Differences.

14. In an arithmetical progression, any term after the first is formed by adding a constant difference to the preceding term.

If, therefore, each term of an A. P. be subtracted from the following term, we obtain a series of differences each equal to the common difference. If each term of this series be subtracted from the following term, we obtain a series of differences each equal to zero.

Thus,

$$\begin{array}{c} 1,\ 3,\ 5,\ 7,\ 9,\ \cdots, \\ 2,\ 2,\ 2,\ 2,\ \cdots, \\ 0,\ 0,\ 0,\ \cdots. \end{array}$$

15. An arithmetical progression is a particular instance of a more general class of series, from which, by continuing to form series of differences as in Art. 14, a series of differences each equal to zero is finally obtained.

Thus

Given series,	1,	4,	9,	16,	25,	⋯,
1st differences,	3,	5,	7,	9,	⋯,	
2d differences,	2,	2,	2,	⋯,		
3d differences,	0,	0,	⋯.			

16. The first series of differences is sometimes called the **First Order of Differences**; the second series of differences, the **Second Order of Differences**; and so on.

The *n*th Term of the Series.

17. To find any term of an arithmetical progression, it is sufficient to have the first term of the given series and the first term of the first order of differences (the common difference).

To find any term of the more general series, it will prove to be sufficient to have the first term of the given series and the first term of each order of differences.

Observe that any term of the given series is found by adding to the preceding term the corresponding term of the first order of differences. Thus, in the example of Art. 15,

$$4 = 1 + 3,\ \ 9 = 4 + 5,\ \ 16 = 9 + 7,\ \cdots.$$

In like manner, any term of any order of differences is formed by adding to the preceding term the corresponding term of the next order of differences. Thus, in the first order of differences,

$$5 = 3 + 2,\ \ 7 = 5 + 2,\ \ 9 = 7 + 2,\ \cdots.$$

Now, let a_1 be the first term of the given series,

d_1 be the first term of the first order of differences,

d_2 be the first term of the second order of differences,

and so on.

First, forming the second terms of the series as indicated above, then the third terms from the second terms, and so on, we have:

Given series, $a_1,\ \ a_1+d_1,\ \ a_1+2\,d_1+d_2,\ \ a_1+3\,d_1+3\,d_2+d_3,\ \cdots,$

1st order of dif., $d_1,\ \ d_1+d_2,\ \ d_1+2\,d_2+d_3,\ \ \cdots,$

2d order of dif., $d_2,\ \ d_2+d_3,\ \ \cdots,$

3d order of dif., $d_3,\ \ \cdots.$

We now have

$$a_1 = a_1,\quad a_2 = a_1 + d_1,\quad a_3 = a_1 + 2\,d_1 + d_2, \text{ etc.}$$

Observe that the numerical coefficients in the expression for a_2 are the same as in $x + y$, those in the expression for a_3, the same as in the expansion of $(x+y)^2$, etc. That is, in the formula for each of the first four terms, the numerical coefficients are the same as in the expansion of a power of a binomial whose *degree is one less than the number of the term.*

We will now prove that, if a similar formula hold to any term, it holds to the next term. Let us assume that the formula holds to the nth term.

Then, $$a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{\lfloor 2}d_2 + \cdots + \frac{(n-1)(n-2)\cdots(n-r)}{\lfloor r}d_r + \cdots.$$

Now the first order of differences is a series of the same character as the given series. If, therefore, the above formula hold for the nth term of the given series, it holds for the nth term of the first order of differences. Let ${}_nd_1$ stand for this term.

Then, $${}_nd_1 = d_1 + (n-1)d_2 + \frac{(n-1)(n-2)}{\lfloor 2}d_3 + \cdots + \frac{(n-1)(n-2)\cdots(n-r+1)}{\lfloor r-1}d_r + \cdots.$$

But $$a_{n+1} = a_n + {}_nd_1.$$

Therefore, a_{n+1}

$$= a_1 + nd_1 + \frac{n(n-1)}{\lfloor 2}d_2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}d_r + \cdots.$$

Hence, if the formula hold to the nth term, it holds to the $(n+1)$th. But, as was first proved, it holds to the fourth term; therefore, it holds to the fifth; and so on.

We therefore have

$$\boldsymbol{a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{\lfloor 2}d_2 + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}d_3 + \cdots.}$$

Ex. Find the 15th term of the series

$$1 + 4 + 9 + 17 + 29 + \cdots.$$

We have
$$\begin{array}{llllll} a_1 = 1, & 4, & 9, & 17, & 29, & \cdots, \\ d_1 = 3, & 5, & 8, & 12, & \cdots, & \\ d_2 = 2, & 3, & 4, & \cdots, & & \\ d_3 = 1, & 1, & \cdots, & & & \\ d_4 = 0, & \cdots. & & & & \end{array}$$

Then,
$$a_{15} = 1 + 14 \cdot 3 + \frac{14 \cdot 13}{\lfloor 2} \cdot 2 + \frac{14 \cdot 13 \cdot 12}{\lfloor 3} = 589.$$

The Sum of n Terms of the Series.

18. Let it be required to find the sum of n terms of the series
$$a_1 + a_2 + a_3 + a_4 + \cdots a_n + \cdots.$$

We form a new series of which the first term is 0, the second a_1, the third $a_1 + a_2$, the fourth $a_1 + a_2 + a_3$, and so on. The $(n+1)$th term of the assumed series is evidently the required sum of n terms of the given series.

We then have

$$\begin{array}{l} 0,\quad a_1,\quad a_1 + a_2,\quad a_1 + a_2 + a_3,\quad a_1+a_2+a_3+a_4,\quad \cdots, \\ \quad a_1,\quad a_2,\qquad a_3,\qquad\qquad a_4,\quad \cdots, \\ \qquad d_1,\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \\ \qquad\quad d_2,\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \\ \qquad\qquad d_3 \cdots. \end{array}$$

Since the first order of differences of the new series is the same as the given series, the second order of differences of the new series is the same as the first order of differences of the given series, and so on.

Observe that 0 is taken as the first term of the assumed series in order that the first order of differences may begin with a_1 and therefore the first terms of the orders of differences following be in order d_1, d_2, d_3, $\cdots$.

We now find the $(n + 1)$th term of the assumed series, and take the result as the sum of n terms of the given series. In applying the formula of the preceding article, we must let

$$a_1 = 0,\ d_1 = a_1,\ d_2 = d_1,\ \cdots.$$

If S_n stand for the required sum, we have

$$S_n = na_1 + \frac{n(n-1)}{\underline{|2}}d_1 + \frac{n(n-1)(n-2)}{\underline{|3}}d_2 + \cdots,$$

wherein a_1, d_1, d_2 $\cdots$, have the same meanings, with reference to the given series, as in the preceding article.

Ex. Find the sum of the squares of the first n natural numbers
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.$$

We have

$$a_1 = 1,\quad 4,\quad 9,\quad 16,\quad \cdots,$$
$$d_1 = 3,\quad 5,\quad 7,\quad \cdots,$$
$$d_2 = 2,\quad 2,\quad \cdots,$$
$$d_3 = 0,\quad \cdots.$$

Then, $$S_n = n + \frac{n(n-1)}{\lfloor 2} \cdot 3 + \frac{n(n-1)(n-2)}{\lfloor 3} \cdot 2$$

$$= \tfrac{1}{6}[6n + 9n(n-1) + 2n(n-1)(n-2)] = \tfrac{1}{6}n(n+1)(2n+1).$$

Piles of Cannon-balls.

19. Cannon-balls, oranges, and other spherical objects are usually piled in the form of pyramids, consisting of horizontal layers.

(i.) **Square Pyramids.** — When the base of the pyramid is a square, the top layer evidently contains 1 ball, the second layer 2^2 balls, the third layer 3^2 balls, and so on.

Therefore, the number of balls in a square pyramid of n layers is the sum of n terms of the series

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots.$$

Hence, by Ex., Art. 18, $\boldsymbol{S_n = \tfrac{1}{6}n(n+1)(2n+1)}$.

Observe that n is also the number of balls in a side of the bottom layer.

(ii.) **Triangular Pyramids.** — When the base of the pyramid is an equilateral triangle, the top layer contains 1 ball, the second layer 3 balls, the third layer 6 balls, and so on.

Hence, the number of balls in a triangular pyramid of n layers is the sum of n terms of the series

$$1 + 3 + 6 + 10 + \cdots.$$

By Art. 18, we have

$$\boldsymbol{S_n} = n + \frac{n(n-1)}{\lfloor 2} \cdot 2 + \frac{n(n-1)(n-2)}{\lfloor 3} = \boldsymbol{\tfrac{1}{6}n(n+1)(n+2)},$$

wherein n is also the number of balls in a side of the bottom layer.

(iii.) **Rectangular Pyramids.** — In a rectangular pyramid, the first layer consists of a row of balls, the second layer of two rows, the third layer of three rows, and so on.

If the first layer contain k balls, the second layer contains $2(k+1)$, the third $3(k+2)$, and so on.

Hence, the number of balls in n layers is the sum of n terms of the series

$$k+2(k+1)+3(k+2)+4(k+3)+\cdots.$$

We have $a_1=k,\ d_1=k+2,\ d_2=2,\ d_3=0,\ \cdots.$

Therefore,

$$S_n=nk+\frac{n(n-1)}{\lfloor 2}(k+2)+\frac{n(n-1)(n-2)}{\lfloor 3}\cdot 2$$

$$=\tfrac{1}{6}n(2n^2+3nk+3k-2)=\tfrac{1}{6}n(n+1)(2n+3k-2).$$

If m be the number of balls in the length of the bottom layer, then

$$m=k+n-1,\ \text{ whence }\ k=m-n+1,$$

wherein n is also the number of balls in the breadth of the bottom layer. The preceding formula may now be written

$$\boldsymbol{S_n=\tfrac{1}{6}n(n+1)(3m-n+1).}$$

Ex. 1. Find the number of balls in a rectangular pyramid, the sides of the bottom layer containing 14 and 20 balls respectively.

Since the number of layers is the same as the number of balls in the breadth of the bottom layer, we have

$$S_{14}=\tfrac{1}{6}\cdot 14\cdot 15\cdot(60-14+1)=1645.$$

Ex. 2. Find the number of oranges in an incomplete triangular pyramid, if the number in a side of the top layer be 5 and the number in a side of the bottom layer be 10.

The pyramid, if complete, would consist of ten layers, and hence would contain $\frac{1}{6}\cdot 10\cdot 11\cdot 12,=220$, oranges.

The part which is wanting would be a pyramid of four layers, and hence would contain $\frac{1}{6}\cdot 4\cdot 5\cdot 6,=20$, oranges. Therefore, the required number of oranges is

$$220-20,=200.$$

EXERCISES III.

1. Find the ninth term and sum of the first nine terms of the series $1, 4, 8, 13, \cdots$.

2. Find the twelfth term and the sum of the first fifteen terms of the series, $3, 2, 3, 6, \cdots$.

3. Find the fourteenth term and the sum of the first twenty terms of the series, $5, 3, 4, 7, 11, \cdots$.

4. Find the eighth term and the sum of the first nine terms of the series

$$2a - 3b, 3a - 2b, 4a - 2b, 5a - 3b, \cdots.$$

Find the nth term and the sum of the first n terms each of the series:

5. $2, 6, 16, 32, \cdots$.

6. $5, 3, 13, 35, \cdots$.

7. $5, 1, -6, -14, -21, \cdots$.

8. $6, 3, -4, -13, -22, \cdots$.

9. $a, 2a + 1, 3a + 3, 4a + 6, \cdots$.

10. Find the sum of the cubes of the first n natural numbers.

Find the number of balls in

11. A square pyramid, having 12 balls in each side of the base.

12. A triangular pyramid, having 15 balls in each side of the base.

13. A rectangular pyramid, having 12 and 5 balls respectively in the length and breadth of the base.

Find the number of balls required to complete

14. A square pyramid, having 144 balls in the top layer.

15. A triangular pyramid, having 5 balls in a side of the top layer.

16. A rectangular pyramid, having 22 and 8 balls respectively in the length and breadth of the top layer.

17. How many balls in an incomplete rectangular pyramid, having 16 and 12 balls respectively in the length and breadth of the top layer, and 25 in the breadth of the bottom layer?

18. How many layers in a square pyramid, containing 91, $= 7 \cdot 13$, balls?

19. A rectangular pyramid of 8 layers contains 1860 balls. How many balls are in the top row?

20. The number of balls in a square pyramid, increased by 91, is equal to twice the number in a triangular pyramid of the same number of layers. How many layers in each pyramid?

21. Show that the number of balls in any square pyramid is one-fourth of the number in a triangular pyramid having twice as many layers.

Interpolation.

20. The following example will indicate the nature of an important application of the method of finite differences.

Ex. Given $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $\cdots$, find the value of $(2\frac{1}{2})^2$.

We have $a_1 = 1$, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, $\cdots$.

Since the first term is the square of 1, the second term the square of 2, and so on, we may call the square of $2\frac{1}{2}$ the $2\frac{1}{2}$th term. Then,

$$a_{2\frac{1}{2}} = 1 + \tfrac{3}{2} \cdot 3 + \frac{\frac{3}{2} \cdot \frac{1}{2}}{2} \cdot 2 = \tfrac{25}{4},$$

which agrees with the result obtained by squaring.

21. Interpolation is the process of inserting between the terms of a given series other terms which conform to the law of the series.

In thus interpolating terms in a given series it is necessary, as we have seen, to give to n a fractional value in the formula for the nth term. Interpolation is extensively applied in astronomy, and is also used in computing numbers intermediate between those given in mathematical tables.

Ex. 1. From a table of square roots we obtain $\sqrt{4} = 2$, $\sqrt{5} = 2.2361$, $\sqrt{6} = 2.4495$, $\sqrt{7} = 2.6458$, $\sqrt{8} = 2.8284$; find $\sqrt{5.25}$.

We have

$$\begin{array}{l} a_1 = 2.0000,\quad 2.2361,\quad 2.4495,\quad 2.6458,\quad 2.8284,\quad \cdots, \\ \quad d_1 = .2361,\quad .2134,\quad .1963,\quad .1826,\quad \cdots, \\ \quad\quad d_2 = -.0227,\quad -.0171,\quad -.0137,\quad \cdots, \\ \quad\quad\quad d_3 = .0056,\quad .0034,\quad \cdots, \\ \quad\quad\quad\quad d_4 = -.0022,\quad \cdots. \end{array}$$

It should be kept in mind that in forming differences we always subtract a number from the number on its right, thereby sometimes obtaining negative remainders.

Since $\sqrt{5}$ is the second term and $\sqrt{6}$ is the third term, for $\sqrt{5.25}$ we take $n = 2.25, = \frac{9}{4}$.

Then,
$$\begin{aligned} a_{\frac{9}{4}} &= 2 + \tfrac{5}{4}(.2361) + \frac{\frac{5}{4}\cdot\frac{1}{4}}{2!}(-.0227) \\ &\quad + \frac{\frac{5}{4}\cdot\frac{1}{4}(-\frac{3}{4})}{3!}(.0056) + \frac{\frac{5}{4}\cdot\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{4!}(-.0022) \\ &= 2 + .2951 - .0035 - .0002 - .00003 \\ &= 2.2914. \end{aligned}$$

In such examples it is not possible to obtain a series of differences whose terms are zero. But, by taking more terms in the given series, more orders of differences, with terms nearer to zero, can be obtained. The results are approximate, and the work is to be carried only so far as it will affect the last decimal place in the values of the given terms.

EXERCISES IV.

1. Given $\log 30 = 1.47712$, $\log 31 = 1.49136$, $\log 32 = 1.50515$, $\log 33 = 1.51851$, $\log 34 = 1.53148$; find $\log 31.8$.

2. Given $\sqrt[3]{9} = 2.0801$, $\sqrt[3]{10} = 2.1544$, $\sqrt[3]{11} = 2.2240$, $\sqrt[3]{12} = 2.2894$, $\sqrt[3]{13} = 2.3513$; find $\sqrt[3]{11.25}$.

3. The latitude of a place on the earth's surface is obtained by observing the altitude of the sun at noon, and adding to the complement of the altitude the sun's declination. On Oct. 31, 1896, the altitude of the sun at Philadelphia was found to be

35° 36′ 37″.3. The Nautical Almanac gives the following values of the sun's declination:

Oct. 30, 1896,	at $15^m\ 29^s$ past 7 A.M.,	—14° 3′ 3″.8;
31,	" " " " "	—14° 22′ 28″.7;
Nov. 1,	" " " " "	—14° 41′ 39″.8;
2,	" " " " "	—15° 0′ 36″.6.

Find the latitude of the place of observation.

4. The length of a degree of longitude on the equator is 69.17 miles; at 10° N. latitude, 68.13 miles; at 20°, 65.03 miles; at 30°, 59.96 miles; at 40°, 53.06 miles; at 50°, 44.55 miles. What is the length of a degree of longitude at 36° 15′ N. latitude?

5. The altitude of a star as seen with the eye is greater than it really is, on account of the refraction of the rays of light from the star by the earth's atmosphere. The altitude of the star Fomalhaut was observed to be 19° 15′ 37″.3. Find its correct altitude, using the following table:

Altitude.	Refraction.	Altitude.	Refraction.
16°	3′ 18″.4	22°	2′ 21″.9
18°	2′ 55″.8	24°	2′ 8″.9
20°	2′ 37″.3	26°	1′ 57″.9

6. The temperature of a litre of water is 23°.4 C. Find its weight, using the following table of densities, the unit being the density of water at the temperature of maximum density:

Temperature.	Density.	Temperature.	Density.
10° C.	.99974	25° C.	.99713
15° C.	.99915	30° C.	.99577
20° C.	.99827		

7. Nitric acid is diluted with water so that the solution contains 22.4% acid. Find the weight of a litre of the solution, using the following table of specific gravities:

% of Acid.	Sp. gr.	% of Acid.	Sp. gr.
0	.999	20	1.121
10	1.058	30	1.187

CHAPTER XXXII.

THE EXPONENTIAL AND LOGARITHMIC SERIES.

1. In this chapter we shall give two important series, and from them derive formulæ for computing naperian and common logarithms.

The Exponential Series.

2. The expression a^x, in which the variable enters as an exponent, is called an **Exponential Function**.

3. Let $\qquad y = a^x$, then $\log_e y = x \log_e a, \qquad (1)$

wherein e is the base of the naperian system of logarithms.

From (1), we obtain $y = e^{x \log_e a}$, or, $a^x = e^{x \log_e a}. \qquad (2)$

If, therefore, we expand $e^{x\log_e a}$ in a series, to ascending powers of x, the result will be also the expansion of a^x.

4. If $n > 1$, the expansion of $\left(1 + \frac{1}{n}\right)^n$ by the Binomial Theorem will be a convergent series, by Ch. XXVII., Art. 8.

Therefore,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{\underline{|3}}\cdot\frac{1}{n^3}$$
$$+\cdots+\frac{n(n-1)\cdots(n-r+1)}{\underline{|r}}\cdot\frac{1}{n^r}+\cdots,$$
$$= 1 + 1 + \frac{1-\frac{1}{n}}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \cdots$$
$$+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{\underline{|r}}+\cdots,$$

however great n may be. If then we let n increase indefinitely,

$$\frac{1}{n} \doteq 0,\ \frac{2}{n} \doteq 0, \text{ etc.}$$

Hence, when $n \doteq \infty$, $\left(1+\frac{1}{n}\right)^n \doteq 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots+\frac{1}{\lfloor r}+\cdots.$

Denoting the series $1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots$ by the letter e, as in Ch. XXVIII., Art. 12, we have

$$\lim_{n \doteq \infty}\left(1+\frac{1}{n}\right)^n = e.$$

5. It follows from the result of the preceding article that the expansion of e^x can be obtained from the expansion of $\left[\left(1+\frac{1}{n}\right)^n\right]^x$, by letting n increase indefinitely.

We have $\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}$

$$= 1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\cdot\frac{1}{n^3}$$
$$+\cdots+\frac{nx(nx-1)\cdots(nx-r+1)}{\lfloor r}\cdot\frac{1}{n^r}+\cdots$$

$$= 1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}$$
$$+\cdots+\frac{x\left(x-\frac{1}{n}\right)\cdots\left(x-\frac{r-1}{n}\right)}{\lfloor r}+\cdots.$$

Letting n increase indefinitely, we have

$$e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots+\frac{x^r}{\lfloor r}+\cdots. \qquad (1)$$

This series is found, by Ch. XXV., Art. 21, to be convergent for all finite values of x.

Therefore, replacing x by $x\log_e a$, and remembering that $e^{x\log_e a} = a^x$, we have

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{2!} + \frac{x^3(\log_e a)^3}{3!} + \cdots. \qquad (2)$$

This **Exponential Series** is convergent for all finite values of x.

The Logarithmic Series.

6. From the expansion of a^x in Art. 5, we have

$$a^x - 1 = x\log_e a + \frac{x^2(\log_e a)^2}{2!} + \frac{x^3(\log_e a)^3}{3!} + \cdots.$$

Dividing by x, $\dfrac{a^x-1}{x} = \log_e a + \dfrac{x(\log_e a)^2}{2!} + \dfrac{x^2(\log_e a)^3}{3!} + \cdots.$

The series in the second member is convergent for all finite values of x. Therefore, by Ch. XXVI., Art. 3, its sum approaches $\log_e a$, as $x \doteq 0$.

Whence, $$\lim_{x \doteq 0} \frac{a^x-1}{x} = \log_e a;$$

or, replacing x by α, $$\lim_{\alpha \doteq 0} \frac{a^\alpha-1}{\alpha} = \log_e a. \qquad (1)$$

7. Substituting $1+x$ for a in the relation of the preceding article, we have

$$\log_e(1+x) = \lim_{\alpha \doteq 0} \frac{(1+x)^\alpha - 1}{\alpha}. \qquad (1)$$

If $x < 1$ numerically, the expansion of $(1+x)^\alpha$ by the Binomial Theorem is a convergent series.

Therefore,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots,$$

and from (1),

$$\log_e(1+x) = \lim_{\alpha \doteq 0}\left[x + \frac{\alpha-1}{2!}x^2 + \frac{(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots\right]$$

whence, $$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \qquad (2)$$

It is important to keep in mind that this expansion holds only when x lies between -1 and $+1$. It is therefore of little practical value in computing logarithms. A series which can be used for this purpose will be derived in the next article.

8. Replacing x by $-x$ in (2), Art. 7,

we have $$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots. \qquad (1)$$

Subtracting (1) from (2), Art. 7,

$$\log_e(1+x) - \log(1-x),\ = \log_e\frac{1+x}{1-x},\ = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right). \qquad (2)$$

In (2), let $\frac{1+x}{1-x} = \frac{1+n}{n}$, whence $x = \frac{1}{2n+1}$.

Then $\log_e \frac{1+n}{n}$,

$$= \log_e(1+n) - \log_e n,\ = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$$

or

$$\boldsymbol{\log_e(1+n) = \log_e n + 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]} \qquad (3)$$

The series (2) is convergent when $x < 1$, and positive. Therefore the series (3) is convergent when $\frac{1}{2n+1} < 1$; or $n > o$.

Hence, this series is equal to $\log_e(1+n)$ for all positive values of n, however great.

Computation of Logarithms.

9. Naperian Logarithms. — By means of the formula derived in the preceding article, the naperian logarithms of all positive numbers can be computed.

Let $n = 1$. Then,

$$\log_e 2 = \log_e 1 + 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \cdots\right)$$
$$= 0 + 2\,(.3333333 + .0123457 + .0008231 + .0000653$$
$$+ .0000057 + .0000005)$$
$$= .69315, \text{ to five places of decimals.}$$

Letting $n = 2, 4, 6, \cdots$, we have

$$\begin{aligned}
\log_e\ 3 &= \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \cdots\right) &&= 1.09861.\\
\log_e\ 4 &= 2 \log_e 2 &&= 1.38629.\\
\log_e\ 5 &= \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \cdots\right) &&= 1.60944.\\
\log_e\ 6 &= \log_e 2 + \log_e 3 &&= 1.79176.\\
\log_e\ 7 &= \log_e 6 + 2\left(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5} + \cdots\right) &&= 1.94591.\\
\log_e\ 8 &= 3 \log_e 2 &&= 2.07944\\
\log_e\ 9 &= 2 \log_e 3 &&= 2.19722\\
\log_e 10 &= \log_e 2 + \log_e 5 &&= 2.30259
\end{aligned}$$

and so on.

It is necessary to compute only logarithms of prime numbers, by the formula. Logarithms of composite numbers are obtained by adding together the logarithms of their factors.

As the numbers increase, fewer terms in the formula are required to compute their logarithms to a given decimal place.

10. Common Logarithms. — Let $n = \log_{10} N$, or $10^n = N$.

Taking logarithms, to base e, $n \log_e 10 = \log_e N$.

Whence, $$n = \frac{1}{\log_e 10} \cdot \log_e N,$$

or $\log_{10} N = \dfrac{1}{\log_e 10} \cdot \log_e N, = \dfrac{1}{2.30259} \cdot \log_e N, = .43429 \log_e N$

The number

$$\frac{1}{\log_e 10} = \frac{1}{2.30259} = .43429,$$

is called the **Modulus** of the common system with respect to the Naperian system.

Hence, to compute the common logarithm of any positive number, multiply the naperian logarithm by the modulus of the common system, .43429.

E.g., $\log_{10} 2 = .43429 \times \log_e 2, \quad = .43429 \times .69315, \quad = .30103.$
Only common logarithms of prime numbers should be thus obtained. Common logarithms of composite numbers are obtained by adding together the logarithms of their factors.

11. Evidently, the naperian logarithm of a number can be obtained from the common logarithm by dividing by .43429, or by multiplying by 2.30259.

The number 2.30259 is called the **Modulus** of the naperian system with respect to the common system.

EXERCISES.

1–10. Compute the common logarithms of all integers from 11 to 20 inclusive.

11. Find the sum of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$

12. Find the sum of $\frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3^2} + \frac{1}{3} \cdot \frac{1}{3^3} - \cdots.$

Expand, to ascending powers of x:

13. $\frac{1}{2}(e^{ix} + e^{-ix}).$ **14.** $\frac{1}{2i}(e^{ix} - e^{-ix}).$

15. Prove $\lim_{x \doteq \infty}\left(1 + \frac{m}{x}\right)^{\frac{x}{n}} = e^{\frac{m}{n}}.$

16. Show that
$$\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots$$
$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots.$$

17. Show that $\frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \cdots = \log_e a - \log_e b.$

18. Show that, when $x > 1$,
$$\log\sqrt{x^2 - 1} = \log x - \left(\frac{1}{2x^2} + \frac{1}{4x^4} + \frac{1}{6x^6} + \cdots\right).$$

CHAPTER XXXIII.

DETERMINANTS.

1. The equations $\qquad a_1x + a_2y = 0, \qquad (1)$

$$b_1x + b_2y = 0, \qquad (2)$$

have evidently the solution 0, 0. Let us inquire if they can be simultaneously satisfied by values of x and y other than 0, 0.

Multiplying (1) by b_2, and (2) by a_2,

$$a_1b_2x + a_2b_2y = 0,$$

$$a_2b_1x + a_2b_2y = 0.$$

Subtracting, $\qquad (a_1b_2 - a_2b_1)x = 0.$

This equation will be satisfied by a value of x other than 0, if $\qquad a_1b_2 - a_2b_1 = 0.$

The same result would have been obtained, had we first eliminated y.

2. The expression $\qquad a_1b_2 - a_2b_1 \qquad$ (I.)

is an example of a form which occurs frequently in mathematics, and for which a special notation has been devised.

Such an expression is called a **Determinant**.

The determinant (I.) is usually written in a square form:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

The *positive term* of the determinant, obtained from the square form, is the cross-product from upper left to lower right; the *negative term* is the cross-product from upper right to lower left.

E.g., $\begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} = 4\cdot 5 - 2\cdot 3 = 14$; $\begin{vmatrix} 3 & -1 \\ 2 & 7 \end{vmatrix} = 3\cdot 7 - (-1)2 = 23.$

3. We shall frequently call the symbolic form in which the determinant is written the determinant.

The advantage of writing determinants in this form will be made evident in subsequent work.

4. The **Elements** of a determinant are the unconnected symbols in its square form; as a_1, a_2, b_1, b_2 in (I.).

A **Row** of a determinant is a horizontal line of elements; as a_1, a_2.

A **Column** of a determinant is a vertical line of elements; as $\begin{matrix} a_1 \\ b_1 \end{matrix}$.

The rows are numbered *first*, *second*, etc., counting from top to bottom; and the columns from left to right.

The square form has two *diagonals*.

The **Principal Diagonal** is composed of the elements from the upper left-hand corner to the lower right-hand corner; as $\begin{matrix} a_1 & \\ & b_2 \end{matrix}$ in (I.).

The **Order** of a determinant is the number of rows or columns in its square form.

Thus, $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is a determinant of the second order.

EXERCISES I.

Find the values of the determinants:

1. $\begin{vmatrix} 5 & 3 \\ 8 & 5 \end{vmatrix}$. **2.** $\begin{vmatrix} -9 & 8 \\ -7 & 6 \end{vmatrix}$. **3.** $\begin{vmatrix} 11 & 0 \\ -5 & 7 \end{vmatrix}$. **4.** $\begin{vmatrix} 4 & 8 \\ 5 & 10 \end{vmatrix}$.

5. $\begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix}$. **6.** $\begin{vmatrix} b & 2a \\ 2c & b \end{vmatrix}$. **7.** $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$.

Verify the following identities.

8. $\begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix} = 0$. **9.** $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. **10.** $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$.

11. $\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0$. **12.** $\begin{vmatrix} ka_1 & ka_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 \\ kb_1 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

13. $\begin{vmatrix} a_1 + \alpha_1 & a_2 \\ b_1 + \beta_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_2 \\ \beta_1 & b_2 \end{vmatrix}$.

5. Next, let us inquire when the equations

$$a_1x + a_2y + a_3z = 0, \tag{1}$$

$$b_1x + b_2y + b_3z = 0, \tag{2}$$

$$c_1x + c_2y + c_3z = 0, \tag{3}$$

are satisfied by values of x, y, z, other than 0, 0, 0.

Multiplying (2) by c_3, and (3) by b_3,

$$b_1c_3x + b_2c_3y + b_3c_3z = 0,$$

$$b_3c_1x + b_3c_2y + b_3c_3z = 0.$$

Subtracting, $(b_1c_3 - b_3c_1)x + (b_2c_3 - b_3c_2)y = 0.$

Whence $$y = -\frac{b_1c_3 - b_3c_1}{b_2c_3 - b_3c_2}\cdot x.$$

In like manner, from (2) and (3), we obtain

$$z = \frac{b_1c_2 - b_2c_1}{b_2c_3 - b_3c_2}\cdot x.$$

Substituting these values of y and z in (1),

$$a_1x - a_2\cdot\frac{b_1c_3 - b_3c_1}{b_2c_3 - b_3c_2}\cdot x + a_3\cdot\frac{b_1c_2 - b_2c_1}{b_2c_3 - b_3c_2}\cdot x = 0,$$

or, $$[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]\,x = 0.$$

This equation will be satisfied by a value of x other than 0, if the expression

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1), \tag{II.}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, \tag{III.}$$

be equal to zero.

The expression (II.) is called a determinant of the third order, and is usually written in the square form

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{IV.}$$

Minors.

6. If the row and column in which a_1 stands, in the determinant (IV.), be deleted, the remaining elements constitute a determinant of the second order, $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$.

This determinant is called the **Minor** of the given determinant with respect to a_1, and may be denoted by A_1.

If the row and column in which a_2 stands be deleted, the remaining elements constitute the determinant $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$.

This determinant, *with sign changed,* is called the minor with respect to a_2, and is denoted by A_2.

In like manner the minor with respect to a_3 is $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$, and is denoted by A_3.

7. The form of the determinant of the third order given in (III.), Art. 5, can now be written

$$a_1A_1 + a_2A_2 + a_3A_3.$$

Written in this form, the determinant is said to be expanded in terms of the elements of the first row and the corresponding minors. By this expansion the value of any determinant of the third order can be readily obtained.

E.g.,

$$\begin{vmatrix} 2 & -3 & 1 \\ 5 & 4 & -2 \\ -1 & 6 & 7 \end{vmatrix} = 2\begin{vmatrix} 4 & -2 \\ 6 & 7 \end{vmatrix} - (-3)\begin{vmatrix} 5 & -2 \\ -1 & 7 \end{vmatrix} + \begin{vmatrix} 5 & 4 \\ -1 & 6 \end{vmatrix}$$
$$= 2(28 + 12) + 3(35 - 2) + (30 + 4) = 213.$$

8. The determinant of the third order can also be expanded in terms of the elements of any row, or column, and the corresponding minors.

For, rearranging the terms of the determinant (II.), Art. 5, we have

$$-b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1), \qquad \text{(V.)}$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2), \text{etc.} \qquad \text{(VI.)}$$

In (V.) and (VI.), $-(a_2c_3 - a_3c_2)$ is the minor with respect to b_1 and is denoted by B_1, etc.

The above expansions can now be written

$$b_1B_1 + b_2B_2 + b_3B_3, \quad a_1A_1 + b_1B_1 + c_1C_1, \text{ etc.}$$

In general, the minor with respect to any element is, *except for sign*, the determinant obtained by deleting the row and column in which the element stands.

The signs to be prefixed to the determinants thus obtained alternate + and −, beginning with a_1, and going either horizontally or vertically, but not diagonally. Thus, the sign of the minor of a_1 is +, of b_1 is −, of b_2 is +, of b_3 is −, of c_3 is +, etc.

Ex. Find the value of $\begin{vmatrix} 4 & -1 & 2 \\ 3 & 0 & 5 \\ 1 & 6 & 7 \end{vmatrix}$.

Since the second row contains a zero element, the work is simplified by expanding in terms of the elements of this row:

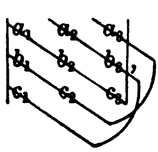
$$-3\begin{vmatrix} -1 & 2 \\ 6 & 7 \end{vmatrix} + 0\begin{vmatrix} 4 & 2 \\ 1 & 7 \end{vmatrix} - 5\begin{vmatrix} 4 & -1 \\ 1 & 6 \end{vmatrix} = 57 + 0 - 125 = -68.$$

9. The terms of a *determinant of the third order* can also be obtained directly from the square form. Removing parentheses in (II.), Art. 5, and writing positive terms first, we have

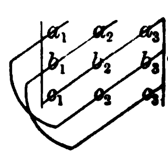
$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

The terms are obtained by multiplying together the elements connected by continuous lines in the following schemes:

POSITIVE TERMS



NEGATIVE TERMS



Ex. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8 - 1\cdot6\cdot8 - 2\cdot4\cdot9 - 3\cdot5\cdot7 = 0.$

Principles of Determinants.

10. The form in (VI.), Art. 8 can be regarded as the expansion

either of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, or of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

That is, *the value of a determinant remains the same when the rows are changed into columns, and the columns into rows.*

11. *If all the elements of any row, or column, of a determinant be* 0, *the determinant vanishes.*

For, expanding in terms of the zero elements, we have

$$\begin{vmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - 0\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + 0\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = 0.$$

12. *If any two rows, or columns, of a determinant be interchanged, the sign of the determinant is changed, but its numerical value remains the same.*

For, $\quad a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$$= -[b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1),$$

or $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

That is, the principle enunciated holds when two adjacent rows are interchanged. To interchange the first and third rows is equivalent to interchanging the first and second rows, then in the resulting determinant the second and third rows, and finally in the last determinant the second and first rows.

These transformations give

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = - \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}.$$

13. *If two rows, or two columns, of a determinant be identical, the determinant vanishes.*

Let D be the value of the determinant. If the two identical rows, or columns, be interchanged, the value of the resulting determinant will be $-D$, by the preceding article. But this interchange of identical rows, or columns, does not alter the given determinant.

Hence $\qquad D = -D$, and therefore $D = 0$.

14. Let

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3, \qquad (1)$$

$$= a_1A_1 + b_1B_1 + c_1C_1, \text{ etc.} \qquad (2)$$

Then $\qquad b_1A_1 + b_2A_2 + b_3A_3$

differs from (1) only in having b's where (1) has a's. That is,

$$b_1A_1 + b_2A_2 + b_3A_3 = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

In like manner, $a_2A_1 + b_2B_1 + c_2C_1 = 0$, etc.

15. *If each element of a row, or column, of a determinant be multiplied by the same number, the determinant is multiplied by that number.*

For,

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = ka_1A_1 + ka_2A_2 + ka_3A_3$$

$$= k(a_1A_1 + a_2A_2 + a_3A_3).$$

Hence the truth of the principle enunciated.

It follows that, if the elements of any row, or column, of a determinant be the same multiples of the corresponding elements of any other row, or column, the determinant vanishes.

16. *If each element of any row, or column, of a determinant be the sum of two terms, the determinant can be expressed as the sum of two determinants.*

For,
$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= (a_1+\alpha_1)A_1 + (a_2+\alpha_2)A_2 + (a_3+\alpha_3)A_3$$
$$= (a_1A_1 + a_2A_2 + a_3A_3) + (\alpha_1A_1 + \alpha_2A_2 + \alpha_3A_3)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

In like manner, if each element of any row, or column, be the sum of m terms, the determinant can be expressed as the sum of m determinants.

By a proof similar to that above, we have

$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1+\beta_1 & b_2+\beta_2 & b_3+\beta_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

If the elements of the first two rows be sums of m and n terms respectively, the determinant can be expressed as the sum of mn terms. In like manner, the principle can be extended to a determinant in which the elements of all the rows are sums of any numbers of terms.

17. *If like multiples of the elements of any row, or column, be added to, or subtracted from, the corresponding elements of any other row, or column, the value of the determinant is not changed.*

For, $\begin{vmatrix} a_1 + kb_1, & a_2 + kb_2, & a_3 + kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= (a_1A_1 + a_2A_2 + a_3A_3) + k(b_1A_1 + b_2A_2 + b_3A_3),$ by Art. 16,

$= a_1A_1 + a_2A_2 + a_3A_3,$ since $b_1A_1 + b_2A_2 + b_3A_3 = 0$, by Art. 14.

The purpose of this principle is to reduce the elements of a row, or column, to smaller numbers, as many as possible to zero, before expanding the determinant.

Ex. 1. Find the value of $\begin{vmatrix} 1 & -4 & 2 \\ 5 & 3 & 9 \\ -6 & 8 & -7 \end{vmatrix}$.

We have

$$\begin{vmatrix} 1 & -4 & 2 \\ 5 & 3 & 9 \\ -6 & 8 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 5 & 23 & 9 \\ -6 & -16 & -7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 5 & 23 & -1 \\ -6 & -16 & 5 \end{vmatrix} = \begin{vmatrix} 23 & -1 \\ -16 & 5 \end{vmatrix} = 99.$$

To obtain the second determinant we multiply the elements of the first column by 4, and add the products to the corresponding elements of the second column; to obtain the third determinant we multiply the elements of the first column of the second determinant by 2, and subtract the products from the corresponding elements of the third column. The third determinant is then expanded in terms of the elements of the first row.

With a little practice we can replace two or more rows, or columns, at the same time. Thus, the third determinant could have been obtained directly from the first, by writing the one result of the two consecutive steps.

Ex. 2. Find the value of $\begin{vmatrix} -4 & 8 & 15 \\ 6 & -5 & -4 \\ 14 & 11 & 9 \end{vmatrix}$.

We have

$$\begin{vmatrix} -4 & 8 & 15 \\ 6 & -5 & -4 \\ 14 & 11 & 9 \end{vmatrix} = 2\begin{vmatrix} -2 & 8 & 15 \\ 3 & -5 & -4 \\ 7 & 11 & 9 \end{vmatrix} = 2\begin{vmatrix} -2 & 8 & 15 \\ 1 & 3 & 11 \\ 7 & 11 & 9 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 14 & 37 \\ 1 & 3 & 11 \\ 0 & -10 & -68 \end{vmatrix} = -2\begin{vmatrix} 14 & 37 \\ -10 & -68 \end{vmatrix}$$

$$= -2\begin{vmatrix} 4 & -31 \\ -10 & -68 \end{vmatrix} = 1164.$$

To obtain the second determinant, the factor 2 is removed from the elements of the first column. The elements of the first row of the second determinant are added to the corresponding elements of the second row, thus introducing an element 1 in the first column of the third determinant. To obtain the fourth determinant, the elements of the second row are multiplied by 2 and added to the corresponding elements of the first row, and by 7 and subtracted from the corresponding elements of the third row. In the fifth determinant the elements of the second row are added to the corresponding elements of the first row, to introduce smaller numbers before expanding.

It is important to notice that the row, or column, which is replaced is the row, or column, that we *add to*, or *subtract from*.

Thus, multiplying the elements of the first column of the determinant $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$ by 2, and adding to the corresponding elements of the second column, we have

$$\begin{vmatrix} 1 & 0 \\ 3 & 10 \end{vmatrix}, = 10.$$

Had we multiplied and added as above, but replaced the elements of the first column by the sums, we should have obtained

$$\begin{vmatrix} 0 & -2 \\ 10 & 4 \end{vmatrix}, = 20$$

18. The work of evaluating a determinant can often be simplified by some special device.

Ex. 1. Find the value of $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

If we let $a = b$, the resulting determinant will have two rows identical, and will therefore vanish. Hence, $a - b$ is a factor of the determinant. For similar reasons $b - c$ and $c - a$ are factors. The product of these three factors is of the same degree as the determinant. Therefore this product can differ from the value of the determinant only in a numerical factor. Assume the value of the determinant to be

$$k(a-b)(b-c)(c-a).$$

If we identify one term of the determinant with the corresponding term of this product, all terms will be identical. The term obtained from the principal diagonal is bc^2; the corresponding term in the product is kbc^2. Hence $k = 1$.

Therefore $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

2. Find the value of $\begin{vmatrix} a & b+c & a^2 \\ b & a+c & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

Adding the elements of the second column to the corresponding elements of the first column, we have

$$\begin{vmatrix} a+b+c & b+c & a^2 \\ a+b+c & a+c & b^2 \\ a+b+c & a+b & c^2 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b+c & a^2 \\ 1 & a+c & b^2 \\ 1 & a+b & c^2 \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & b+c & a^2 \\ 0 & a-b & b^2-a^2 \\ 0 & a-c & c^2-a^2 \end{vmatrix} = (a-b)(a-c)(a+b+c)\begin{vmatrix} 1 & -(a+b) \\ 1 & -(a+c) \end{vmatrix}$$

$$= (a-b)(a-c)(b-c)(a+b+c).$$

Ex. 3. Solve the equation $\begin{vmatrix} a & a & x \\ k & k & k \\ b & x & b \end{vmatrix} = 0.$

Factoring out k, we have

$$\begin{vmatrix} a & a & x \\ 1 & 1 & 1 \\ b & x & b \end{vmatrix} = \begin{vmatrix} a & 0 & x-a \\ 1 & 0 & 0 \\ b & x-b & 0 \end{vmatrix} = -\begin{vmatrix} 0 & x-a \\ x-b & 0 \end{vmatrix} = (x-a)(x-b).$$

Therefore $(x-a)(x-b)=0$; whence $x=a$, $x=b$.

EXERCISES II.

Find the values of the determinants:

1. $\begin{vmatrix} 1 & 2 & -3 \\ 5 & 4 & -10 \\ -7 & 10 & 11 \end{vmatrix}$. 2. $\begin{vmatrix} 8 & 11 & 14 \\ 9 & 12 & 15 \\ 10 & 13 & 16 \end{vmatrix}$. 3. $\begin{vmatrix} 11 & 12 & 5 \\ -9 & 4 & -3 \\ 2 & -7 & 8 \end{vmatrix}$.

4. $\begin{vmatrix} 19 & -12 & 7 \\ 9 & 18 & -21 \\ -11 & -24 & 49 \end{vmatrix}$. 5. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. 6. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

7. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. 8. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$. 9. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix}$.

10. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$. 11. $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$.

12. $\begin{vmatrix} a & a^2-bc & a^2 \\ b & b^2-ca & b^2 \\ c & c^2-ab & c^2 \end{vmatrix}$. 13. $\begin{vmatrix} a+b & a+c & b+c \\ b & a & c \\ a-b & a-c & b-c \end{vmatrix}$.

Solve the following equations:

14. $\begin{vmatrix} 10-3x & 4 & 3 \\ 2-2x & -2 & 4 \\ 11-x & 9 & -5 \end{vmatrix} = 0.$ 15. $\begin{vmatrix} a-bx & b-cx & c-ax \\ a+b & b+c & a+c \\ 1 & 1 & 1 \end{vmatrix} = 0.$

16. $\begin{vmatrix} 3-x & 7+2x & 11-3x \\ 5+x & 3-2x & 17+3x \\ 1 & 2 & 3 \end{vmatrix}=0.$ **17.** $\begin{vmatrix} 2x-1 & 3x+2 & -7 \\ 3x+2 & 5x-4 & -9 \\ 4x+8 & 7x+1 & -20 \end{vmatrix}=0$

Solution of Linear Simultaneous Equations.

19. Solve the equations

$$a_1x + a_2y + a_3z = a_4, \tag{1}$$

$$b_1x + b_2y + b_3z = b_4, \tag{2}$$

$$c_1x + c_2y + c_3z = c_4. \tag{3}$$

Let $$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

the determinant whose elements are, in order, the coefficients of the unknown numbers in the given equations.

Multiplying (1) by A_1, (2) by B_1, and (3) by C_1, we have

$$a_1A_1x + a_2A_1y + a_3A_1z = a_4A_1, \tag{4}$$

$$b_1B_1x + b_2B_1y + b_3B_1z = b_4B_1, \tag{5}$$

$$c_1C_1x + c_2C_1y + c_3C_1z = c_4C_1. \tag{6}$$

Adding (4)–(6), we obtain

$$(a_1A_1 + b_1B_1 + c_1C_1)x + (a_2A_1 + b_2B_1 + c_2C_1)y$$
$$+ (a_3A_1 + b_3B_1 + c_3C_1)z = a_4A_1 + b_4B_1 + c_4C_1.$$

Now, the coefficient of x is D, the coefficients of y and z vanish, by Art. 14, and the second member is the determinant

$$\begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix}. \quad \text{Therefore,} \quad x = \frac{\begin{vmatrix} a_4 & a_2 & a_3 \\ b_4 & b_2 & b_3 \\ c_4 & c_2 & c_3 \end{vmatrix}}{D}.$$

In like manner, we can obtain

$$y = \frac{\begin{vmatrix} a_1 & a_4 & a_3 \\ b_1 & b_4 & b_3 \\ c_1 & c_4 & c_3 \end{vmatrix}}{D}, \qquad z = \frac{\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}}{D}.$$

Notice that *the common denominator of these values of x, y, z, is the determinant of the coefficients of x, y, z, in the given equations.*

In the value of x, the determinant in the numerator is obtained from that in the denominator by replacing the column of coefficients of x in the given equations by the column of second members.

In general, *the determinant in the numerator of the value of any unknown number is obtained from the common denominator by replacing the column of coefficients of that unknown number by the column of second members in the given equations;* as in the values of y and z.

Ex. Solve the equations

$$3x + 25y - 6z = 35,$$

$$6x - 10y + 21z = 49,$$

$$8x + 15y - 14z = -4.$$

We have

$$x = \frac{\begin{vmatrix} 35 & 25 & -6 \\ 49 & -10 & 21 \\ -4 & 15 & -14 \end{vmatrix}}{\begin{vmatrix} 3 & 25 & -6 \\ 6 & -10 & 21 \\ 8 & 15 & -14 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 3 & 35 & -6 \\ 6 & 49 & 21 \\ 8 & -4 & -14 \end{vmatrix}}{\begin{vmatrix} 3 & 25 & -6 \\ 6 & -10 & 21 \\ 8 & 15 & -14 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 3 & 25 & 35 \\ 6 & -10 & 49 \\ 8 & 15 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 25 & -6 \\ 6 & -10 & 21 \\ 8 & 15 & -14 \end{vmatrix}}.$$

Whence $x = 1$, $y = 2$, $z = 3$.

Determinants of Higher Order.

20. The principles given in the preceding articles hold also for determinants of order higher than the third. The student is referred to treatises on determinants for a fuller development of the subject.

The following example will illustrate the evaluation of a determinant of the fourth order.

Ex. Find the value of $\begin{vmatrix} 13 & 0 & 3 & 14 \\ 3 & -2 & 1 & 4 \\ -2 & 5 & 2 & 9 \\ 18 & -3 & 5 & 22 \end{vmatrix}$.

Subtracting three times the elements of the third column from the corresponding elements of the first column, adding twice the elements of the third column to the corresponding elements of the second column, and subtracting four times the elements of the third column from the corresponding elements of the fourth column, we have

$$\begin{vmatrix} 4 & 6 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ -8 & 9 & 2 & 1 \\ 3 & 7 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 6 & 2 \\ -8 & 9 & 1 \\ 3 & 7 & 2 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ -8 & 9 & 1 \\ 3 & 7 & 2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 2 & 3 & 1 \\ -10 & 6 & 0 \\ -1 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} -10 & 6 \\ -1 & 1 \end{vmatrix} = 8.$$

The second determinant above is obtained from the first by expanding the latter in terms of the elements of the second row.

In general, such an expansion of a determinant of the fourth order gives four determinants of the fourth order. Since each of the latter contains six terms, the general determinant of the fourth order contains twenty-four terms.

Observe that a determinant of the fourth order cannot be expanded by a method similar to that given in Art. 9, for expanding a determinant of the third order. For, by a similar method we should obtain only eight terms.

21. The following considerations, which also hold for determinants of any order, enable us to write a particular term, with its proper sign.

We have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

(i.) *Each term contains one element from each row and one element from each column.*

An *inversion* of order occurs in a line of numbers when one of them is out of its natural order with respect to some other number. For example, in 132 there is one inversion of order, since 3 is out of order with respect to 2; in 231 there are two inversions, since 2 and 3 are out of order with respect to 1.

We determine the sign of any particular term as follows:

(ii.) *Write the letters in the order in which they indicate the rows in the given determinant, as in the above expansion.*

Prefix the sign + to any term, if the number of inversions of order in the subscripts be even, the sign —, if the number of inversions be odd.

Thus, in the term $a_2b_1c_3$, there is one inversion in the subscripts. Therefore the sign of this term is —, as in the above expansion. In like manner, the signs of the other terms are determined.

It is important to note that the natural order in the subscripts is the order in which they denote the columns in the given determinant, which may or may not be the natural order of numbers.

EXERCISES III.

Find the values of the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$. **2.** $\begin{vmatrix} 4 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 \\ 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 4 \end{vmatrix}$. **3.** $\begin{vmatrix} 1 & 7 & 5 & 8 \\ 2 & 7 & 3 & 6 \\ 3 & 15 & 9 & 4 \\ 4 & 15 & 7 & 2 \end{vmatrix}$.

4. $\begin{vmatrix} 8 & 11 & 12 & 7 \\ 5 & -9 & 7 & 6 \\ 3 & 7 & -13 & 2 \\ 2 & -5 & 4 & 3 \end{vmatrix}.$ **5.** $\begin{vmatrix} 29 & -18 & 15 & 13 \\ -14 & -17 & 21 & 5 \\ 4 & 19 & 11 & -7 \\ 5 & -6 & 7 & -8 \end{vmatrix}.$

6. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$ **7.** $\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & x^2 & y^2 \\ b^2 & x^2 & 0 & z^2 \\ c^2 & y^2 & z^2 & 0 \end{vmatrix}.$ **8.** $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$

Solve the following systems of equations:

9. $\begin{cases} 50x - 23y = 27, \\ 48x + 25y = 73. \end{cases}$ **10.** $\begin{cases} 55x + 43y = -19, \\ 17x - 11y = 67. \end{cases}$

/ **11.** $\begin{cases} x + y + z = 14, \\ 6x - 3y - 7z = 0, \\ 4x - 9y + 7z = 0. \end{cases}$ / **12.** $\begin{cases} 2x + 3y - 4z = 1, \\ 3x + 4z - 5y = 2, \\ 4y + 5z - 6x = 3. \end{cases}$

13. $\begin{cases} 8x + 21y - 9z = -23, \\ 12x - 28y - 15z = 89, \\ 15x - 49y + 18z = 29. \end{cases}$ / **14.** $\begin{cases} 21x + 15y + 11z = 84, \\ 5x - 7y + 10z = 21, \\ 8x + 9y - 7z = 5. \end{cases}$

15. $\begin{cases} 15x + 17y + 13z = 90, \\ 17x + 13y + 15z = 90, \\ 13x + 15y + 17z = 90. \end{cases}$ **16.** $\begin{cases} x + y + z = 1, \\ ax + by + cz = n, \\ a^2x + b^2y + c^2z = n^2. \end{cases}$

17. $\begin{cases} ax + by + cz = a, \\ bx + cy + az = b, \\ cx + ay + bz = c. \end{cases}$ **18.** $\begin{cases} x + ay + a^2z + a^3 = 0, \\ x + by + b^2z + b^3 = 0, \\ x + cy + c^2z + c^3 = 0. \end{cases}$

19. $\begin{cases} x + 2y = 5, \\ 2y + 3z = 13, \\ 3z + 4u = 25, \\ 4u + 5x = 21. \end{cases}$ **20.** $\begin{cases} x + y + z + u = 1, \\ ax + by + cz + du = n, \\ a^2x + b^2y + c^2z + d^2u = n^2. \\ a^3x + b^3y + c^3z + d^3u = n^3. \end{cases}$

CHAPTER XXXIV.

THEORY OF EQUATIONS.

General Form.

1. If all the terms of a rational integral equation of the nth degree in x be brought to the first member and terms containing like powers of x be united, we have the equivalent equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0. \quad \text{(I.)}$$

In this equation the coefficients a_0, a_1, a_2, $\cdots$, a_n do not contain x, but otherwise may denote any numbers, real or imaginary. Any of the a's, except a_0, may be 0.

A **Complete Equation** of the nth degree is one which contains all powers of the unknown number from the 0th to the nth inclusive.

E.g., $x^3 - 3x^2 + 6x + 1 = 0$ is a complete cubic equation.

An **Incomplete Equation** of the nth degree is one in which some powers of the unknown number are wanting.

E.g., $2x^3 - 5x - 5 = 0$ is an incomplete cubic equation, since the term in x^2 is wanting.

2. The first member of equation (I.) is a function of x by Ch. XXIV., Art. 3. For the sake of brevity in subsequent work, we shall denote this function of x by $f(x)$; that is,

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

For example, if $f(x) = x^2 - 5x + 6$, then $f(x) = 0$ is only an abbreviated statement for $x^2 - 5x + 6 = 0$. (1)

The roots of equation (1) are 2 and 3, since

$$2^2 - 5 \times 2 + 6 = 0, \text{ and } 3^2 - 5 \times 3 + 6 = 0.$$

These statements, in the abbreviated notation, are

$$f(2) = 0, \text{ and } f(3) = 0.$$

In general, if r be a root of $f(x) = 0$, then $f(r) = 0$.

Synthetic Division.

3. In the following work we shall have frequent occasion to divide $f(x)$ by expressions of the form $x - r$.

Ex. 1. Divide $3x^3 - 4x^2 + 7x - 3$ by $x - 2$.

We know that the quotient will be of the second degree. Let us assume it with undetermined coefficients to be

$$b_0x^2 + b_1x + b_2.$$

The division can be continued until the remainder is free of x. Let this remainder be denoted by R. Then, by Ch. III., Art. 48, we have

$$\begin{aligned}3x^3 - 4x^2 + 7x - 3 &= (x-2)(b_0x^2 + b_1x + b_2) + R\\ &= b_0x^3 + (b_1 - 2b_0)x^2 + (b_2 - 2b_1)x + (R - 2b_2).\end{aligned}$$

Whence, by Ch. XXVI., Art. 3,

$$\left.\begin{array}{rcr} b_0 = 3, & \text{and} & b_0 = 3;\\ b_1 - 2b_0 = -4, & & b_1 = 2b_0 - 4 = 2;\\ b_2 - 2b_1 = 7, & & b_2 = 2b_1 + 7 = 11;\\ R - 2b_2 = -3, & & R = 2b_2 - 3 = 19. \end{array}\right\} \quad (2)$$

The quotient is $3x^2 + 2x + 11$, and the remainder is 19.

From the relations (2), we see that b_0 is the same as the coefficient of x^3 in the given function; that b_1 is obtained by multiplying b_0 by 2, and adding the product to the second coefficient in the given function; and so on. The following arrangement, in which only coefficients are written, will be seen to be equivalent to the above:

$$\begin{array}{rrrrl} 3 & -4 & 7 & -3 & (\underline{2}\\ & 6 & 4 & 22 & \\ \hline 3 & 2 & 11 & 19 & \end{array}$$

This example illustrates the following method: *Write the coefficients of the given expression in a horizontal line and the number which is subtracted from x in the divisor on its right, as 2 above.* This number is called the **Synthetic Divisor**.

Bring down the first coefficient as the coefficient of the first term of the quotient; multiply this coefficient by the synthetic divisor and add the product to the second given coefficient; multiply this sum by the synthetic divisor and add the product to the third given coefficient; and so on.

The last sum will be the remainder, and the preceding numbers in order the coefficients of the quotient.

If any term be wanting in the given function, it must be supplied with a zero coefficient.

Ex. 2. Divide $x^4 - 3x^2 + 2x - 5$ by $x - 3$.

We have

$$\begin{array}{rrrrr|l} 1 & 0 & -3 & 2 & -5 & 3 \\ & 3 & 9 & 18 & 60 & \\ \hline 1 & 3 & 6 & 20 & 55 & \end{array}$$

The quotient is $x^3 + 3x^2 + 6x + 20$, and the remainder is 55.

Ex. 2. Divide $2x^3 + 5x^2 - 4x - 5$ by $x + 2, = x - (-2)$.

Observe that the synthetic divisor is -2.

$$\begin{array}{rrrr|l} 2 & 5 & -4 & -5 & -2 \\ & -4 & -2 & 12 & \\ \hline 2 & 1 & -6 & 7 & \end{array}$$

The quotient is $2x^2 + x - 6$, and the remainder is 7.

The Remainder Theorem.

4. If $3x^3 - 4x^2 - 6x + 7$ be divided by $x - 2$, we obtain a partial quotient $3x^2 + 2x - 2$, and a remainder 3.

If, now, 2 be substituted for x in the given expression, we have $3 \times 2^3 - 4 \times 2^2 - 6 \times 2 + 7, = 3$, the above remainder.

This example illustrates the following principle:

If a rational integral expression in x be divided by $x - r$, the remainder of the division will be equal to the result of substituting r for x in the given expression.

Let $$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

be the given expression.

If Q stand for the quotient of the division by $x-r$, and R for the remainder, we have, by Ch. III., Art. 48,

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = Q(x-r) + R.$$

Substituting r for x in the last equation, we obtain

$$a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = Q(r-r) + R, = R.$$

5. Observe that the process of synthetic division explained in Art. 3 may now be regarded also as a process of substituting the synthetic divisor for x in the given function.

Ex. Substitute -3 for x in $4x^3 - 3x^2 + 7x + 1$.

The result of the substitution will be equal to the remainder obtained by dividing $4x^3 - 3x^2 + 7x + 1$ by $x+3$.

$$\begin{array}{rrrr|l} 4 & -3 & 7 & 1 & \underline{-3} \\ & -12 & 45 & -156 & \\ \hline 4 & -15 & 52 & -155 & \end{array}$$

The result of the substitution is -155.

6. *If r be a root of $f(x)=0$, then $f(x)$ is exactly divisible by $x-r$; and, conversely, if $f(x)$ be exactly divisible by $x-r$, then r is a root of $f(x)=0$.*

For, if r be a root of function of $f(x)=0$, then, $f(r)=0$. Therefore, by the preceding article, $R=0$.

Also, if $R=0$, then $f(r)=0$, and r is a root of $f(x)=0$

EXERCISES I.

By the synthetic method, divide

1. $x^3 - 3x^2 - 10x + 24$ by $x-2$; by $x-3$; by $x+3$.

2. $4x^3 - 24x^2 + 13x + 35$ by $x+4$; by $x-5$; by $x+7$.

3. $x^4 + 6x^3 - 22x + 15$ by $x-8$; by $x+5$; by $x-4$.

4. $32x^4 - 46x^2 + 9x + 5$ by $x+2$; by $x-\frac{1}{2}$; by $x+\frac{3}{4}$.

Substitute for x, by the synthetic method, in $x^3-8x^2+17x-4$ the following numbers:

5. 2. **6.** -2. **7.** 3. **8.** -3. **9.** 4.

In $2x^4 - 9x^3 - 17x^2 - 6$ substitute the following numbers:

10. 4. **11.** -4. **12.** 5. **13.** -5. **14.** 6.

Number of Roots.

7. As we have seen, every equation of the first degree in one unknown number has one and only one root; every equation of the second degree in one unknown number has two and only two roots. If we assume that every equation has a root, real or imaginary, we can prove that a rational integral equation of the nth degree has n, and only n, roots. The proof that every equation has a root is too difficult for the scope of this book.

Let r_1 be a root of $f(x)=0$.

Then, by Art. 6, $\qquad f(x)=(x-r_1)\,Q_1,$

wherein Q_1 is a rational integral function of the $(n-1)$th degree in x.

Since every equation has a root, let r_2 be a root of $Q_1=0$.

Then, $$Q_1=(x-r_2)\,Q_2,$$

wherein Q_2 is of the $(n-2)$th degree.

Continuing in this way, we finally obtain a quotient of degree 1, $=n-(n-1)$, in x. This quotient is therefore Q_{n-1}.

Let r_n be a root of $Q_{n-1}=0$.

Then, $$Q_{n-1}=(x-r_n)\,Q_n,$$

wherein Q_n is free of x.

Evidently the first term in Q_1 is a_0x^{n-1}, in Q_2 is a_0x^{n-2}, and so on.

The first term in Q_{n-1} is a_0x, and hence $Q_n=a_0$.

Therefore $$Q_{n-1}=(x-r_n)a_0.$$

We now have
$$\begin{aligned} f(x) &= (x-r_1)Q_1,\\ &= (x-r_1)(x-r_2)Q_2,\\ &= \cdots\cdots\cdots\cdots\\ &= (x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)a_0. \end{aligned}$$

We know that r_1 is a root of $f(x)=0$. Now, let $x=r_2$.

Then, $f(r_2)=a_0\,(r_2-r_1)\,0\,(r_2-r_3)\cdots(r_2-r_n)=0.$

Therefore, r_2 is a root of $f(x)=0$.

In like manner it can be shown that $r_3,\ r_4,\ \cdots\ r_n$, are roots.

Let k be any number different from $r_1,r_2,\cdots,r_n$.

Then, $f(k)=(k-r_1)(k-r_2)\cdots(k-r_n)$.

Since k is not equal to any r, the factors on the right are all

different from 0, and therefore their product does not reduce to 0; hence k is not a root of $f(x) = 0$.

Therefore, *if an equation of the nth degree have one root, it has n, and only n, roots.*

Equal Roots.

8. If two or more of the r's be equal, the equation has *equal* roots.

Ex. $x^3 - 5x^2 + 8x - 4 = (x-1)(x-2)(x-2)$.

The roots are 1, 2, 2. In such cases the equation is said still to have n roots.

Depression of Equations.

9. It follows from Art. 7 that if one root of an equation be known, the remaining roots can be obtained as follows:

Divide the given equation by x minus the known root.

The quotient, equated to zero, is called the **Depressed Equation.**

Solve the depressed equation.

Ex. One root of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 1.

Dividing the given equation by $x - 1$, we obtain the depressed equation $x^2 - 5x + 6 = 0$. The roots of this equation are 2 and 3. Therefore, the three roots of the given equation are 1, 2, 3.

EXERCISES II.

Solve the following equations:

1. $x^3 + x^2 - 17x + 15 = 0$, one root being 3.
2. $x^3 - 8x^2 + 25x - 50 = 0$, one root being 5.
3. $9x^3 + 18x^2 - 4x - 8 = 0$, one root being -2.
4. $4x^3 + 12x^2 - 11x + 20 = 0$, one root being -4.
5. $4x^3 - 9x^2 + 14x - 3 = 0$, one root being $\frac{1}{4}$.
6. $27x^3 - 63x - 34 = 0$, one root being $-\frac{2}{3}$.
7. $x^4 + 3x^3 - 19x^2 - 27x + 90 = 0$, two roots being 2, -5.
8. $x^4 - x^3 - 17x^2 + 5x + 60 = 0$, two roots being 4, -3.
9. $12x^4 - 59x^3 + 33x^2 + 99x + 27 = 0$, two roots being 3, 3.
10. $x^4 - 20x^2 - 21x - 20 = 0$, two roots being 5, -4.

Relation between the Roots and Coefficients.

10. We can, without loss of generality, divide both members of an equation by the coefficient of x^n. We thus obtain an equation of the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

wherein $p_1, p_2, \cdots, p_n$ denote any numbers, real or imaginary.

11. If r_1, r_2, r_3 be the roots of the equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0,$$

we have

$$\begin{aligned} x^3 + p_1x^2 + p_2x + p_3 &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3. \end{aligned}$$

In general,

$$\begin{aligned} x^n + p_1x^{n-1} &+ p_2x^{n-2} + \cdots + p_{n-1}x + p_n \\ &= (x - r_1)(x - r_2)(x - r_3)\cdots(x - r_n) \\ &= x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} \\ &\quad + (r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)x^{n-2} \\ &\quad - (r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n)x^{n-3} \\ &\quad + \cdot \quad \cdot \quad \cdot \quad + (-1)^n r_1r_2\cdots r_n. \end{aligned}$$

Equating coefficients of like powers of x, we have

$$\begin{aligned} r_1 + r_2 + r_3 + \cdots &= -p_1; \\ r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n &= p_2; \\ r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &= -p_3; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r_1r_2\cdots r_n &= (-1)^n p_n. \end{aligned}$$

From these equations we obtain the following relations between the roots and the coefficients, when the coefficient of the highest power of x is unity:

The sum of the roots is equal to the coefficient of the ~~second term~~ x^{n-1} term *with sign changed.*

The sum of the products of the roots in pairs is equal to the coefficient of the third term.

The sum of the products of the roots taken three at a time is equal to the coefficient of the fourth term with sign changed; and so on.

The product of the roots is equal to the last term if n be even, and to the last term with sign changed if n be odd.

Ex. Given $x^3 - 5x^2 + 3x + 4 = 0$.

The sum of the roots is $-(-5), = 5$.

The sum of the products of the roots, two at a time, is 3.

The product of the roots is -4.

12. The following special cases may be noted:

(i.) *If the sum of the roots be 0, the second term is wanting.*

(ii.) *If one root, or more than one root, be 0, the last term is wanting.*

13. It is important to notice that the relations given in the preceding articles cannot be used to obtain the solution of an equation.

Ex. From the equation $x^3 - 5x^2 + 3x + 4 = 0$, we have

$r_1 + r_2 + r_3 = 5$ (1), $\quad r_1r_2 + r_1r_3 + r_2r_3 = 3$ (2), $\quad r_1r_2r_3 = -4$ (3).

Multiplying (1) by r_1^2, (2) by $-r_1$, (3) by 1, and adding the resulting products, we obtain $r_1^3 - 5r_1^2 + 3r_1 + 4 = 0$.

This is a mere statement that r_1 satisfies the given equation, and in no way helps us to find its value.

14. The properties of Art. 11 can sometimes be used to solve an equation, if a relation between the roots be known.

Ex. One root of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is twice a second root. Solve the equation.

Let $r_2 = 2r_1$. From the relations,

$$r_1 + r_2 + r_3 = 6, \qquad r_1r_2 + r_1r_3 + r_2r_3 = 11, \qquad r_1r_2r_3 = 6,$$

we have

$3r_1 + r_3 = 6$ (1), $\quad 2r_1^2 + 3r_1r_3 = 11$ (2), $\quad r_1^2r_3 = 3$ (3).

From (1) and (2) we obtain $r_1 = 1, \frac{11}{7}$, and $r_3 = 3, \frac{9}{7}$.

Since these values of r_1 and r_3 must also satisfy equation (3), 1, 3, is the only admissible solution.

Finally, $r_2 = 2$, and the required roots are 1, 2, 3.

Symmetrical Functions.

15. An expression is symmetrical with respect to two letters if it remain the same when these letters are interchanged.

E.g., $$ab,\ a+b,\ a^2+2\,ab+b^2$$

are symmetrical with respect to a and b, since when a and b are interchanged they become

$$ba,\ b+a,\ b^2+2\,ba+a^2.$$

An expression is symmetrical with respect to three or more letters if it be symmetrical with respect to any two of them.

E.g., $$r_1r_2+r_1r_3+r_2r_3,\ \ r_1^2+r_2^2+r_3^2,$$

are symmetrical with respect to the three letters r_1, r_2, r_3; for, if any two letters, say r_1 and r_2, be interchanged, we obtain

$$r_2r_1+r_2r_3+r_1r_3,\ \ r_2^2+r_1^2+r_3^2.$$

16. The value of an expression which is symmetrical in the roots can be found without solving the equation.

Ex. If r_1, r_2, r_3, be the roots of the equation

$$x^3+p_1x^2+p_2x+p_3=0,$$

find the value of $$r_1^2+r_2^2+r_3^2.$$

We have

$$(r_1+r_2+r_3)^2=r_1^2+r_2^2+r_3^2+2(r_1r_2+r_1r_3+r_2r_3),$$

or $$(-p_1)^2=r_1^2+r_2^2+r_3^2+2\,p_2;$$

whence $$r_1^2+r_2^2+r_3^2=p_1^2-2\,p_2.$$

Formation of an Equation from its Roots.

17. The relations of Art. 11 enable us to form an equation if its roots be known. We may assume that the coefficient of the highest power of the unknown number is 1.

Ex. 1. Form the equation whose roots are -2, 3, 4.

We have

$r_1+r_2+r_3=5$, the coefficient of x, with sign changed.

$r_1r_2+r_1r_3+r_2r_3=-2$, the coefficient of x; and

$r_1r_2r_3=-24$, the term free from x, with sign changed.

Therefore, the required equation is $x^3-5\,x^2-2\,x+24=0$

Ex. 2. Form the equation whose roots are

$$1+\sqrt{-2},\ 1-\sqrt{-2},\ 1+\sqrt{3},\ 1-\sqrt{3}.$$

The work will be simplified by first forming the quadratic equation whose roots are the conjugate imaginaries, and the quadratic equation whose roots are the conjugate surds, and then multiplying together these two equations. The first of these quadratic equations is $x^2-2x+3=0$;

and the second is $x^2-2x-2=0$.

The required equation of the fourth degree is therefore

$$x^4-4x^3+5x^2-2x-6=0.$$

We could also have formed the equation in Ex. 1 by multiplying together the three linear factors which equated to 0 have as roots the given roots. We should thus have obtained

$$(x+2)(x-3)(x-4)=0, \text{ or } x^3-5x^2-2x+24=0.$$

EXERCISES III.

Solve the following equations:

1. $x^3-3x^2-4x+12=0$, the sum of two roots being 0.

2. $2x^3-11x^2+12x+9=0$, two roots being equal.

3. $x^3-6x^2-x=-6$, one root being 2 greater than a second.

4. $3x^3-25x^2-19x+9=0$, the product of two roots being [illegible] 3

5. $x^3-3x^2-6x+8=0$, the roots being in A. P.

6. $2x^3-9x^2-27x+54=0$, the roots being in G. P.

7. $8x^3-30x^2+27x-5=0$, one root being twice the sum of the other two.

8. $x^4-7x^3+9x^2+7x-10=0$, one root being equal and opposite to a second root.

9. $6x^4-13x^3-18x^2+52x-24=0$, one root being the reciprocal of a second root.

10. $8x^4+30x^3-135x^2-135x+162=0$, the roots being in G. P.

11. $x^4-14x^3+51x^2-14x-80=0$, the roots being in A. P.

Given the equations $x^3+p_1x^2+p_2x+p_3=0$

and $$x^4+p_1x^3+p_2x^2+p_3x+p_4=0,$$

express the following symmetrical functions in terms of the coefficients:

12. $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\cdots.$ **13.** $\dfrac{1}{r_1r_2}+\dfrac{1}{r_1r_3}+\cdots.$ **14.** $r_1^3+r_2^3+\cdots.$

Form the equations whose roots are:

15. $2,\ -3,\ 4.$ **16.** $-1,\ 5,\ \frac{2}{3}.$ **17.** $-6,\ 3\pm\sqrt{3}.$

18. $-5,\ 1\pm\sqrt{-5}.$ **19.** $3,\ 3,\ -2,\ -4.$ **20.** $-1,\ 4,\ \frac{1}{2},\ \frac{1}{3}.$

21. $\pm\sqrt{2},\ 3\pm\sqrt{-2}.$ **22.** $\pm\sqrt{-1},\ -2\pm\sqrt{-3}.$

23. $2,\ 2,\ 1\pm\sqrt{2},\ 2\pm\sqrt{-1}.$

If r_1, r_2, r_3 be the roots of the equation $x^3+p_1x^2+p_2x+p_3=0$, form the equation whose roots are:

24. $r_1^2r_2^2,\ r_1^2r_3^2,\ r_2^2r_3^2.$ **25.** $\dfrac{r_1+r_2}{r_3^2},\ \dfrac{r_1+r_3}{r_2^2},\ \dfrac{r_2+r_3}{r_1^2}.$

Real and Rational Roots.

18. *If $\dfrac{r}{s}$, a rational fraction in its lowest terms, be a root of the equation* $a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0,$ *wherein $a_0, a_1, \cdots, a_n$ are now assumed to be integers, then a_0 is divisible by s, and a_n is divisible by r.*

Since $\dfrac{r}{s}$ is a root, we have

$$a_0\frac{r^n}{s^n}+a_1\frac{r^{n-1}}{s^{n-1}}+\cdots+a_{n-1}\frac{r}{s}+a_n=0.$$

Multiplying by s^{n-1}, we obtain

$$a_0\frac{r^n}{s}+a_1r^{n-1}+a_2r^{n-2}s+\cdots+a_{n-1}rs^{n-2}+a_ns^{n-1}=0, \qquad (1)$$

or $$\frac{a_0r^n}{s}=-\left(a_1r^{n-1}+a_2r^{n-2}s+\cdots+a_{n-1}rs^{n-2}+a_ns^{n-1}\right).$$

The second member of the last equation is integral. Consequently $\dfrac{a_0r^n}{s}$ must reduce to an integer; that is, a_0r^n must be

divisible by s. Since r, and therefore r^n, is not divisible by s, a_0 must be divisible by s.

Now, multiplying (1) by s, and dividing by r, we readily infer that a_n is divisible by r.

Ex. Forming the equation whose roots are $\frac{1}{2}$, $\frac{2}{3}$, 5, we obtain

$$x^3 - \tfrac{37}{6}x^2 + \tfrac{37}{6}x - \tfrac{5}{3} = 0,$$

or

$$6x^3 - 37x^2 + 37x - 10 = 0.$$

We see that 6 is divisible by 2 and 3, the denominators of the two fractional roots, and that 10 is divisible by 1, 2, and 5.

19. *If the coefficient of the highest power of x in a rational integral equation with integral coefficients be unity, the equation cannot have a rational fractional root.*

For, by the preceding article, the denominator of any fractional root must be a factor of this coefficient, and hence be 1.

Surd Roots.

20. *If the coefficients in a rational integral equation be rational, quadratic surds enter in conjugate pairs.*

That is, if $a + \sqrt{b}$ be a root of $f(x) = 0$, then $a - \sqrt{b}$ is also a root.

Divide $f(x)$ by $(x - a - \sqrt{b})(x - a + \sqrt{b})$, and let Q stand for the quotient, and $Rx + S$ for the remainder, if any.

Then, $f(x) = (x - a - \sqrt{b})(x - a + \sqrt{b})Q + Rx + S.$

Substituting $a + \sqrt{b}$ for x, we have

$$f(a + \sqrt{b}) = R(a + \sqrt{b}) + S = Ra + S + R\sqrt{b}.$$

But since $a + \sqrt{b}$ is a root of $f(x) = 0$, $f(a + \sqrt{b}) = 0$,

and therefore, $\quad Ra + S + R\sqrt{b} = 0.$

Whence, by Ch. XV., Art. 37,

$$R\sqrt{b} = 0, \ (1) \quad \text{and} \quad Ra + S = 0. \ (2)$$

From (1), since $b \neq 0$, $R = 0$; and then, from (2), $S = 0$.

Therefore, if $a + \sqrt{b}$ be a root of $f(x) = 0$, we have

$$f(x) = (x - a - \sqrt{b})(x - a + \sqrt{b})Q,$$

and $a - \sqrt{b}$ is also a root.

The quadratic factor $(x-a-\sqrt{b})(x-a+\sqrt{b})$, $=(x-a)^2-b^2$, corresponding to a pair of conjugate surd roots, is rational in a and b.

Imaginary Roots.

21. *If the coefficients of a rational integral equation be real, imaginary or complex roots enter in conjugate pairs.*

That is, if $a+bi$ be a root of $f(x)=0$, then $a-bi$ is also a root. The proof is similar to that given in the preceding article.

The quadratic factor $(x-a-bi)(x-a+bi)$, $=(x-a)^2+b^2$, corresponding to a pair of conjugate complex roots, is real.

Ex. One root of the equation $x^4-5x^3+3x^2+19x-30=0$ is $2+\sqrt{-1}$; solve the equation.

The quadratic factor corresponding to the roots $2+\sqrt{-1}$ and $2-\sqrt{-1}$ is

$$(x-2-\sqrt{-1})(x-2+\sqrt{-1}), =(x-2)^2+1, =x^2-4x+5.$$

Determining the other quadratic factor, we have

$$x^4-5x^3+3x^2+19x-30=(x^2-4x+5)(x^2-x-6).$$

The roots of the equation $x^2-x-6=0$ are found to be -2, 3. Therefore the required roots are -2, 3, $2\pm\sqrt{-1}$.

EXERCISES IV.

Find the factors of the first members of, and hence solve, the following equations. Also solve by Art. 11:

1. $x^3+3x^2+2x+6=0$; one root: $\sqrt{-2}$.
2. $x^3-2x^2-3x+6=0$; one root: $-\sqrt{3}$.
3. $2x^3+2x^2-19x+20=0$; one root: $\frac{1}{2}(3-i)$.
4. $x^3-7x^2+9x+5=0$; one root: $1+\sqrt{2}$.
5. $4x^4-4x^3-23x^2+4x+19=0$; one root: $\frac{1}{2}+\sqrt{5}$.
6. $x^4-8x^3+21x^2-26x+14=0$; one root: $3-\sqrt{2}$.
7. $36x^4-24x^3-143x^2-146x-50=0$; one root: $-\frac{2}{3}-\frac{1}{2}i$.
8. $x^5-16x^3-4x^2-17x-4=0$; two roots: i, $2+\sqrt{5}$.

To transform an Equation into Another whose Roots are Any Multiples of the Roots of the Given Equation.

22. Transform the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

into another equation whose roots are k times the roots of the given equation.

Let y be the unknown number of the new equation.

Then $$y = kx, \text{ or } x = \frac{y}{k},$$

and $$a_0\frac{y^n}{k^n} + a_1\frac{y^{n-1}}{k^{n-1}} + a_2\frac{y^{n-2}}{k^{n-2}} + \cdots + a_{n-1}\frac{y}{k} + a^n = 0.$$

Multiplying by k^n,

$$a_0y^n + ka_1y^{n-1} + k^2a_2y^{n-2} + \cdots + k^{n-1}a_{n-1}y + k^na_n = 0.$$

That is, to form an equation whose roots are k times the roots of a given equation:

Multiply the coefficient of the second term by k; that of the third term by k^2; that of the fourth term by k^3; and so on.

Ex. Form the equation whose roots are 3 times the roots of the equation $x^3 - 2x^2 - x + 6 = 0$.

We have $$y^3 - 3 \cdot 2y^2 - 3^2 y + 3^3 \cdot 6 = 0,$$

or $$y^3 - 6y^2 - 9y + 162 = 0.$$

23. Ex. 1. Form the equation whose roots are ten times the roots of the equation $x^4 - 3x^3 + 8x^2 + 2x - 19 = 0$.

The transformed equation is

$$y^4 - 10 \times 3y^3 + 10^2 \times 8y^2 + 10^3 \times 2y - 10^4 \times 19 = 0,$$

or, $$y^4 - 30y^3 + 800y^2 + 2000y - 190{,}000.$$

To obtain the transformed equation, when the multiplier is 10:

Annex one cipher to the coefficient of the second term; two ciphers to the coefficient of the third term; three ciphers to the coefficient of the fourth term; and so on.

If any term be wanting, it must be supplied mentally with a zero coefficient.

Ex. 2. Form the equation whose roots are ten times the roots of the equation $x^3 - 5x + 6 = 0$.

We have $$y^3 - 500y + 6000 = 0.$$

24. An important application of this transformation is to obtain an equation in which the coefficient of the highest power of the unknown number is 1 and the other coefficients are integral.

Ex. Transform the equation $12x^3 - 5x^2 - \frac{7}{2}x + 1 = 0$ into an equation without fractional coefficients, in which the coefficient of the third power of the unknown number shall be 1.

Dividing by 12, $x^3 - \frac{5}{12}x^2 - \frac{7}{24}x + \frac{1}{12} = 0$.

Forming an equation whose roots are k times the roots of the given equation, we have $y^3 - \frac{5}{12}ky^2 - \frac{7}{24}k^2y + \frac{1}{12}k^3 = 0$.

We find by inspection that 12 is the least value of k which will make all the coefficients integral. Substituting 12 for k, we obtain $y^3 - 5y^2 - 42y + 144 = 0$.

To transform an Equation into Another whose Roots are Those of the Given Equation with Signs Changed.

25. Let $$f(x) = (x - r_1)(x - r_2)\cdots(x - r_n) = 0 \qquad (1)$$
be the given equation. Substituting $-y$ for x, we have
$$\begin{aligned} f(-y) &= (-y - r_1)(-y - r_2)\cdots(-y - r_n) \\ &= (-1)^n(y + r_1)(y + r_2)\cdots(y + r_n) = 0. \qquad (2) \end{aligned}$$

Evidently the roots of equation (2) are $-r_1, -r_2, \cdots, -r_n$.

If the given equation in x be
$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0,$$
the required equation is
$$a_0y^n - a_1y^{n-1} + \cdots + (-1)^{n-1}a_{n-1}y + (-1)^n a_n = 0.$$

That is, to obtain the transformed equation:

Change the sign of every alternate term, beginning with the second; or the signs of the terms containing odd powers of x.

If any term be wanting it must be mentally supplied with a zero coefficient.

Ex. Transform the equation $x^4+5x^2-3x-2=0$ into another whose roots are those of the given equation with signs changed.

We have $$y^4+5y^2+3y-2=0.$$

Descartes' Rule.

26. If the signs of the terms in the equation

$$x^4-3x^3-4x^2+2x-3$$

be arranged in order, we have $+--+-$.

A **Permanence of Sign** is a succession of two like signs, as $--$, above.

A **Variation of Sign** is a succession of two unlike signs, as $+-$ or $-+$, above.

In the above equation there is one permanence of sign and three variations.

27. *If $f(x)$ be multiplied by $x-r$, the number of variations of sign in the product will be at least one greater than the number of variations of sign in $f(x)$.*

The number of variations of sign in

$$x^8+2x^7-5x^6-x^5-4x^4+7x^3-3x^2+4x+1$$

is *four*. Let us multiply this expression by $x-1$.

Grouping in parenthesis all successive terms which have like signs, we have

$$\begin{array}{l}
+(+x^8+2x^7)-(+5x^6+x^5+4x^4)+(+7x^3)-(+3x^2)+(+4x+1)\\
\quad +x\ -1\\
\hline
+(+x^9+2x^8)-(+5x^7+x^6+4x^5)+(+7x^4)-(+3x^3)+(+4x^2+x)\\
\quad(\quad -\ x^8)-(+2x^7-5x^6-x^5)+(+4x^4)-(+7x^3)+(+3x^2-4x)-1\\
\hline
+(+x^9+x^8)-(+7x^7-4x^6+3x^5)+(+11x^4)-(+10x^3)+(+7x^2-3x)-1
\end{array}$$

Observe first that in the given expression, as grouped, the number of variations of sign is one less than the number of groups; in fact, that *in counting variations we need give attention only to the signs preceding the parentheses.*

Also, that the grouping in the partial products and in the result corresponds to that in the given expression, although thereby variations of sign enter in some of the groups.

We see that the number of variations of sign in the signs before the parentheses in the product is one more than in the given expression. That this should be so, can be inferred from the following considerations.

The distribution of signs in the first partial product is evidently the same as in the given expression.

In the second partial product there is no term in x^9. Therefore in the result the sign of the first term in the first parentheses, and therefore the sign before the parentheses, is the same as in the given expression.

The first term in the second parentheses is obtained by multiplying the last term in the first parentheses in the given expression, $+(+2x^7)$, by -1, and is $-(+2x^7)$, of the same sign as the like term in the first partial product. Therefore, in the result the sign of the first term in the second parentheses, and therefore the sign before the parentheses, is the same as in the given expression.

Similar considerations hold for the other groups. Therefore no variation of sign in the signs of the groups of the given expression is lost.

But to the last term in the second partial product there is no corresponding term in the first partial product. Since this term is obtained by multiplying the last term in the last parentheses by -1, its sign will be opposite to that before the last parentheses. Hence at this point one additional variation of sign in the signs of the groups is obtained.

Therefore, the product contains at least this one more variation of sign than the given expression.

Finally, observe that two variations of sign are gained in the second group. Therefore in multiplying the given expression by $x-1$ we gained three variations of sign.

If any term be wanting in the given expression, it should be supplied with a zero coefficient, and be included in the group with the preceding term.

Evidently the reasoning in the preceding example is general, and holds for any rational integral expression in x.

28. Descartes' Rule. — *The number of real positive roots of $f(x)=0$ cannot be greater than the number of variations of sign in $f(x)$; and the number of real negative roots cannot be greater than the number of variations of sign in $f(-x)$.*

Let $F(x)$ stand for the product formed by multiplying together the factors of $f(x)$ corresponding to imaginary and real negative roots, and let $r_1, r_2, r_3, \cdots, r_k$ be the k real positive roots of $f(x)=0$.

Then, $$f(x)=F(x)(x-r_1)(x-r_2)\cdots(x-r_k).$$

Now, $F(x)(x-r_1)$ has at least one more variation of sign than $F(x)$; the product $F(x)(x-r_1)(x-r_2)$ has at least one more variation of sign than $F(x)(x-r_1)$, and therefore at least two more than $F(x)$, and so on. The last product, that is $f(x)$, will therefore have at least k more variations of sign than $F(x)$. Since there will be at least k variations of sign in $f(x)$, the number of real positive roots of $f(x)=0$ cannot be greater than the number of variations of sign.

By Art. 25, the positive roots of $f(-x)=0$ are, with their signs changed, equal to the negative roots of $f(x)=0$. But, by the preceding proof, the number of real positive roots of $f(-x)=0$ cannot exceed the number of variations of sign in $f(-x)$. Hence the number of real negative roots of $f(x)=0$ cannot exceed the number of variations of sign in $f(-x)$.

29. If $f(x)$ be a complete rational integral expression, the variations of sign in $f(-x)$ are evidently permanences of sign in $f(x)$.

Thus, if $$f(x)=4x^5-x^4+3x^3+2x^2-5x-7,$$
then $$f(-x)=4x^5+x^4+3x^3-2x^2-5x+7.$$

The variations of sign, $3x^3-2x^2$ and $-5x+7$, in $f(-x)$, correspond to the permanences of sign, $3x^3+2x^2$ and $-5x-7$, in $f(x)$.

The second part of Descartes' Rule may therefore be stated as follows:

If $f(x)=0$ be a complete rational integral equation, the number of real negative roots cannot be greater than the number of permanences of sign in $f(x)$.

30. From Descartes' Rule, we infer the following:

(i.) *If the signs of the terms of* $f(x)$ *be all positive, the equation* $f(x) = 0$ *does not have a real positive root.*

(ii.) *If* $f(x) = 0$ *be a complete rational integral equation, and the signs of the terms be alternately* $+$ *and* $-$, *the equation does not have a real negative root.*

Limits to the Roots.

31. In determining the real roots of a given equation, the work is often simplified by knowing that the roots cannot be greater or less than a certain number.

A **Superior Limit** to the real roots is a number greater than the greatest root.

An **Inferior Limit** is a number less than the least root.

Various formulæ have been given for finding the limits to the roots, but we can as a rule determine closer limits by inspection than are obtained by using these formulæ. The following examples will illustrate the method of procedure:

Ex. 1. Find a superior limit to the real roots of the equation

$$x^4 + 3x^3 - 13x^2 - 27x + 36 = 0.$$

Grouping terms and factoring, we have

$$x^2(x^2 - 13) + 3x(x^2 - 9) + 36 = 0.$$

It is evident that the first member of this equation will not reduce to zero for any value of x greater than 4. Therefore 4 is a superior limit to the real roots. In the above example the terms are arranged in groups, each having a positive term first. We then determine by inspection what is the smallest value of x which will make all the groups positive.

An inferior limit is obtained by forming the equation whose roots are those of the given equation with signs changed, and finding a superior limit to the real roots of the transformed equation. This number with its sign changed will be an inferior limit to the real roots of the given equation.

Ex. 2. Find an inferior limit to the real roots of the equation

$$x^4 + 3x^3 - 4x^2 - 30x - 36 = 0.$$

Changing the signs of the alternate terms, we have

$$x^4 - 3x^3 - 4x^2 + 30x - 36 = 0,$$

or

$$2x^4 - 6x^3 - 8x^2 + 60x - 72 = 0.$$

Whence $x^3(x - 6) + x^2(x^2 - 8) + 60(x - 1\frac{1}{5}) = 0.$

It is evident that 6 is a superior limit to the real roots of the transformed equation. Therefore -6 is the required inferior limit to the real roots of the given equation.

Where nearly all of the terms are negative, as in Ex. 2, it is necessary to distribute the term containing the highest power of x among the negative terms, multiplying the equation by some number, if the coefficient of this term be unity or less than the number of negative terms.

EXERCISES V.

Transform the following equations into others whose real rational roots are all integers:

1. $x^3 - \frac{2}{3}x^2 + \frac{7}{9}x - \frac{4}{9} = 0.$
2. $x^3 + 5x^2 - \frac{1}{8}x - \frac{47}{4} = 0.$
3. $8x^3 - 14x^2 - 11x + 3 = 0.$
4. $x^4 - \frac{1}{6}x^3 - \frac{1}{18}x^2 + \frac{1}{27}x - \frac{22}{27} = 0.$
5. $12x^4 + 40x^2 - 9x - 14 = 0.$
6. $x^3 + 4.2x^2 - 6.35x - 1.804 = 0.$
7. $x^3 - 2.5x^2 - 4.5x + 9 = 0.$
8. $x^4 + 12.6x^2 - 4.32x + .1539 = 0.$

9–13. Transform the equations in Exs. 1–5 into others whose roots are those of the given equations with signs changed.

Show that the roots of the following equations are all imaginary:

14. $x^4 + 3x^2 + 5 = 0.$
15. $x^6 + 5x^4 + 8x^2 + 10 = 0.$

Determine the least possible number of imaginary roots of the following equations:

16. $x^4 - 9x^3 - 5x - 4 = 0.$
17. $x^5 - 4x^3 + 3x^2 + 1 = 0.$
18. $3x^7 - x^4 + 5x^3 - 9 = 0.$
19. $2x^7 - 3x^5 - 4x^3 - x = 0.$

32. The principles thus far developed enable us to find all the rational roots of a given rational integral equation.

Ex. 1. Solve the equation $x^4 - x^3 - 16x^2 + 4x + 48 = 0$.

By Descartes' Rule this equation cannot have more than two positive roots, or more than two negative roots. We find that 6 is a superior limit, and -4 an inferior limit, to the real roots. Also, since the coefficient of x^4 is unity, the equation cannot have a fractional root. By Art. 11, the integral roots, positive or negative, are factors of 48. These factors, within the limits to the roots, are ± 1, ± 2, ± 3, 4. We now determine by trial which of these factors, if any, satisfy the equation.

Substituting 1 and -1, we find that these numbers are not roots. Substituting 2, the equation is satisfied, and the depressed equation is found to be

$$x^3 + x^2 - 14x - 24 = 0.$$

Substituting -2 in this depressed equation, we find that this factor is a root, and that the second depressed equation is

$$x^2 - x - 12 = 0.$$

The roots of this equation are found to be -3, 4.

Therefore the roots of the given equation are 2, -2, -3, 4.

Ex. 2. Solve the equation $6x^3 + 23x^2 - 6x - 8 = 0$.

We find that 1 is a superior limit, and -5 an inferior limit, to the roots. Therefore the integral roots, if any, are among the numbers -1, -2, -4, and the fractional roots among $\pm \frac{1}{2}$, $\pm \frac{1}{3}$, $\pm \frac{1}{6}$, $\pm \frac{2}{3}$, $\pm \frac{4}{3}$.

We will first try for fractional roots. Substituting $\frac{1}{2}$,

$$\begin{array}{rrrrl} 6 & 23 & -6 & -8 & (\frac{1}{2} \\ & 3 & 13 & \frac{7}{2} & \\ \hline 6 & 26 & 7 & -\frac{9}{2} & \end{array}$$

Therefore $\frac{1}{2}$ is not a root. Substituting $-\frac{1}{2}$,

$$\begin{array}{rrrrl} 6 & 23 & -6 & -8 & (\underline{-\frac{1}{2}} \\ & -3 & -10 & 8 & \\ \hline 6 & 20 & -16 & 0 & \end{array}$$

Therefore $-\frac{1}{2}$ is a root, and the depressed equation is

$$6x^2+20x-16=0, \text{ or } 3x^2+10x-8=0.$$

The roots of this equation are found to be $\frac{2}{3}$, -4.

Therefore the roots of the given equation are $-\frac{1}{2}$, $\frac{2}{3}$, -4.

The above equation could also have been solved as follows: Letting $x=\frac{1}{6}y$, we have, by Art. 22,

$$y^3+23y^2-36y-288=0.$$

This equation can have no fractional roots. The integral roots are found to be $-3, 4, -24$. Then $x, =\frac{1}{6}y, =-\frac{1}{2}, \frac{2}{3}, -4$.

Although by this method we avoid solving for fractional roots, yet the number of integral factors of the last term which it is necessary to test becomes greater.

Newton's Method.

33. The factors of the last term are as a rule more easily tested by the method which follows.

From Art. 3 we have

$$a_0x^3+a_1x^2+a_2x+a_3=b_0x^3+(b_1-b_0r)x^2+(b_2-b_1r)x+(R-b_2r).$$

Now, let r be a root of the given equation.

Then $R=0$, and the form of the equation which we shall use is

$$b_0x^3+(b_1-b_0r)x^2+(b_2-b_1r)x-b_2r=0.$$

The last term is divisible by r, and the quotient is $-b_2$. Adding this quotient to the coefficient of x, we obtain $-b_1r$.

Dividing this sum by r, and adding the quotient, $-b_1$, to the coefficient of x^2, we obtain $-b_0r$.

Dividing this sum by r, and adding the quotient, $-b_0$, to the coefficient of x^3, we obtain a sum 0.

Arranged as in synthetic division,

$$\begin{array}{cccc|l} b_0 & b_1-b_0r & b_2-b_1r & -b_2r & \underline{r} \\ -b_0 & -b_1 & -b_2 & & \\ \hline 0 & -b_0r & -b_1r & -b_2r & \end{array}$$

It is evident from the form of $f(x)$, given in Art. 3, that the above method is general. Observe, also, that the numbers in

the second line of the work above are, in order, the coefficients of the depressed equation with signs changed.

We thus have the following method:

Divide the last term by r and add the quotient to the coefficient of x. Divide this sum by r and add the quotient to the coefficient of x^2. Divide this sum by r and add the quotient to the coefficient of x^3, and so on. The last quotient will be $-b_0$.

If at any stage the division be not exact, this fact at once shows that the number being tested is not a root.

Ex. Solve the equation $x^3-4x^2-9x+36=0$.

Substituting 1 and -1, we find that they are not roots. Testing -2 by Newton's Method,

$$\begin{array}{rrrl} 1 & -4 & -9 & 36(\underline{-2} \\ & & -18 & \\ \hline & & -27 & 36 \end{array}$$

Since -27 is not divisible by 2, 2 is not a root. Testing 3,

$$\begin{array}{rrrl} 1 & -4 & -9 & 36(\underline{3} \\ -1 & 1 & 12 & \\ \hline 0 & -3 & 3 & 36 \end{array}$$

Since the division at each step is exact and the last quotient is -1, 3 is a root, and the depressed equation is

$$x^2-x-12=0.$$

The roots of this equation are -3, 4. Therefore the roots of the given equation are 3, -3, 4.

EXERCISES VI.

Find a real root of each of the following equations, and solve the depressed equation:

1. $x^3-2x^2-13x-10=0$.
2. $x^3-19x+30=0$.
3. $x^3-5x^2+17x-13=0$.
4. $x^3-7x^2+36=0$.
5. $x^3-5x^2+5x+3=0$.
6. $x^3-8x^2+9x+58=0$.
7. $x^3-7x^2+49x+237=0$.
8. $x^3+x^2-61x-205=0$.
9. $x^4+4x^3-34x^2-76x+105=0$.
10. $x^4-15x^2-10x+24=0$.
11. $x^4-2x^3-13x^2+14x+24=0$.
12. $x^4-37x^2-24x+180=0$.

13. $x^4-8x^3+7x^2+40x-60=0$. **14.** $x^4-x^3-11x^2-x-12=0$

15. $x^4-9x^3+15x^2+39x-70=0$. **16.** $x^4-3x^3-2x^2-32=0$.

17. $x^4+5x^3+11x^2+67x+156=0$. **18.** $x^4-26x^2+57x-18=0$.

19. $x^5+6x^4-2x^3-36x^2+x+30=0$.

20. $x^5-11x^4+36x^3-10x^2-148x+240=0$.

21. $x^5-8x^4-6x^3+134x^2-43x-510=0$.

22. $2x^3+x^2-25x+12=0$. **23.** $5x^3+7x^2-46x+24=0$.

24. $6x^4-13x^3-18x^2+7x+6=0$.

25. $12x^4-64x^3+9x^2+53x+10=0$.

26. $96x^4+128x^3-74x^2-63x+18=0$.

Derived Functions.

34. Given $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$, to find the value of $f(x+h)$. We have

$$\begin{aligned} f(x+h) &= a_0(x+h)^n+a_1(x+h)^{n-1}+a_2(x+h)^{n-2} \\ &\qquad +\cdots+a_{n-1}(x+h)+a_n \\ &= a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n \\ &\quad +h[na_0x^{n-1}+(n-1)a_1x^{n-2}+(n-2)a_2x^{n-3}+\cdots+a_{n-1}] \\ &\quad +\frac{h^2}{\underline{|2}}[n(n-1)a_0x^{n-2}+(n-1)(n-2)a_1x^{n-3}+\cdots+a_{n-2}]+\cdots. \end{aligned}$$

The coefficient of h is derived from $f(x)$ as follows:

Multiply each term of $f(x)$ by the exponent of x in that term and subtract 1 from the exponent.

Thus, from a_0x^n we derive na_0x^{n-1}; from a_1x^{n-1} we derive $(n-1)a_1x^{n-2}$; and so on.

On this account the coefficient of h is called the **First Derived Function**, and is represented by $f'(x)$.

Observe that from $a_{n-1}x$ we derive $1\times a_{n-1}x^0, =a_{n-1}$.

From a term without x we obtain 0.

That is, from $a_n=a_nx^0$ we derive $0\times a_nx^{-1}, =0$.

In a similar way the coefficient of $\frac{h^2}{\underline{|2}}$ is derived from the coefficient h; the coefficient of $\frac{h^3}{\underline{|3}}$ from that of $\frac{h^2}{\underline{|2}}$; and so on.

These successive coefficients are therefore called the **Second Derived Function**, the **Third Derived Function**, and so on; and are represented by $f''(x)$, $f'''(x)$, and so on, respectively.

Notice that $f'(x)$ is of degree one lower than $f(x)$; that $f''(x)$ is of degree two lower than $f(x)$; and so on.

We now have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \cdots.$$

In like manner $f(x+h)$ could have been expanded in terms of ascending powers of x:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{\lfloor 2}f''(h) + \cdots.$$

Ex. Given $f(x) = x^3 - 2x^2 + x + 3$, find $f(x+2)$.

We have $$f(x+2) = f(2) + xf'(2) + \frac{x^2}{\lfloor 2}f''(2) + \cdots.$$

In the terms of the second member, 2 is to be substituted for x in $f(x)$, $f'(x)$, and so on. We now have

$$f(x) = x^3 - 2x^2 + x + 3, \quad f'(x) = 3x^2 - 4x + 1,$$
$$f''(x) = 6x - 4, \quad f'''(x) = 6, \quad f^{\text{iv}}(x) = 0, \text{ etc.}$$

Whence $f(2) = 5, \quad f'(2) = 5, \quad f''(2) = 8, \quad f'''(2) = 6.$

Therefore, $\qquad f(x+2) = 5 + 5x + 4x^2 + x^3.$

Multiple Roots.

35. If r be k times a root of $f(x) = 0$, then $x - r$ is k times a factor of $f(x)$, or $(x-r)^k$ is a factor.

Therefore, $\qquad f(x) = (x-r)^k F(x),$

wherein $F(x)$ stands for the product of the remaining factors of $f(x)$, and does not contain the factor $x - r$.

We now have $f(x+h) = (x - r + h)^k F(x+h)$.

Expanding $f(x+h)$ and $F(x+h)$ by the preceding article, and $(x - r + h)^k$ by the binomial theorem, we have

$$f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \cdots$$
$$= [(x-r)^k + k(x-r)^{k-1}h + \cdots][F(x) + hF'(x) + \cdots]$$
$$= (x-r)^k F(x) + h[k(x-r)^{k-1}F(x) + (x-r)^k F'(x)] + \cdots.$$

Equating coefficients of h,

$$f'(x) = k(x-r)^{k-1}F(x) + (x-r)^k F'(x)$$
$$= (x-r)^{k-1}[kF(x) + (x-r)F'(x)].$$

We now see that $f'(x)$ contains $x-r$ as a factor $k-1$ times; therefore, $f'(x)=0$ has $k-1$ roots equal to r.

That is, *if* $f(x)=0$ *have a multiple root, then* $f'(x)=0$ *has the same root repeated one less time,*

36. It follows also from the preceding article that $(x-r)^{k-1}$ is the H. C. F. of $f(x)$ and $f'(x)$. We can therefore find the multiple roots of any given equation, $f(x)=0$, *by finding the roots of the H. C. F. of* $f(x)$ *and* $f'(x)$ *equated to* 0.

If $f(x)$ and $f'(x)$ have no common factor, $f(x)=0$ does not have a multiple root.

Ex. The equation $x^4-9x^3+23x^2-3x-36=0$ has multiple roots. Solve the equation.

We have $\quad f(x) = x^4 - 9x^3 + 23x^2 - 3x - 36,$

and $\quad f'(x) = 4x^3 - 27x^2 + 46x - 3.$

The H. C. F. of $f(x)$ and $f'(x)$ is found to be $x-3$.

Therefore, $(x-3)^2$ is a factor of $f(x)$, and 3 is a root twice of the given equation. Removing the root 3 twice,

$$\begin{array}{rrrrrl} 1 & -9 & 23 & -3 & -36 & (\underline{3} \\ & 3 & -18 & 15 & 36 & \\ \hline 1 & -6 & 5 & 12 & 0 & \\ & 3 & -9 & -12 & & \\ \hline 1 & -3 & -4 & 0 & & \end{array}$$

The final depressed equation is $x^2-3x-4=0$; whence, $x=-1, 4$. The roots of the given equation are $3, 3, -1, 4$.

EXERCISES VII.

Find the successive derived functions of the following:

1. $2x^2-3x+6$. **2.** $x^4+7x^3-5x^2-3x+11$.

Given $f(x)=x^3-9x^2+14x+24$, find

3. $f(x+1)$. **4.** $f(x-2)$. **5.** $f(x-3)$.

Solve the following equations, which have multiple roots:

6. $x^4 + 6x^3 - 14x^2 - 90x + 25 = 0.$

7. $x^4 - 4x^3 - 4x^2 + 16x + 16 = 0.$

8. $x^4 - 6x^3 + 38x^2 - 112x + 104 = 0.$

9. $x^4 + 2x^3 - 3x^2 + 136x + 464 = 0.$

10. $x^5 - x^4 - 11x^3 + 29x^2 - 26x + 8 = 0.$

11. $x^5 + 7x^4 - 6x^3 - 162x^2 - 459x - 405 = 0.$

Graphic Representation.

37. Two perpendicular straight lines divide the plane in which they lie into four parts, called **Quadrants**. Thus, the lines XX', YY', in Fig. 3, divide the plane of the paper into four quadrants:

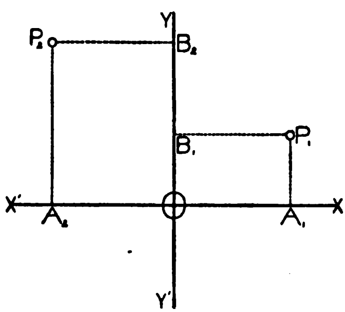


FIG. 3.

XOY called the first quadrant;
YOX' the second quadrant;
$X'OY'$ the third quadrant, and
$Y'OX$ the fourth quadrant.

The position of a point in a plane is known if its distance from these two fixed lines, and the quadrant in which it lies, be known. Thus, the position of a point P_1 in the first quadrant is known, if we know the distances A_1P_1 and B_1P_1; that of the point P_2 in the second quadrant, if we know the distances A_2P_2 and B_2P_2.

38. The two lines $X'OX$ and $Y'OY$ are called **Axes of Reference**, and the point O is called the **Origin**.

The distance of a point P from the axis $Y'OY$, measured along or *parallel to the axis* $X'OX$, is usually designated by the letter x, and is called the **Abscissa**, or the x of the point.

Thus, the x of the point P_1 is $OA_1, = B_1P_1$.

The distance of a point P from the axis $X'OX$, measured along or *parallel to the axis* $Y'OY$, is usually designated by the letter y, and is called the **Ordinate**, or the y of the point.

Thus, the y of the point P_1 is OB_1, $= A_1P_1$.

The abscissa and the ordinate of the point are together called the **Coördinates** of the point.

The axis $X'OX$ is called the **Axis of Abscissas**, or the *x-axis*.

The axis $Y'OY$ is called the **Axis of Ordinates**, or the *y-axis*.

39. The quadrant in which a point lies is determined as follows:

Abscissas measured to the right of the origin are taken positively; those to the left, negatively. Ordinates measured upward are taken positively; those downward, negatively. The signs of the x and y of a point thus determine the quadrant in which it lies.

Locate the point $x = 2$, $y = 3$.

The most convenient way is to measure two units along the x-axis to the right, and from the point thus reached three units upward parallel to the y-axis, as in Fig. 4.

Instead of writing $x = 2$, $y = 3$, we may indicate the coördinates of the point by (2, 3), it being understood that the first number is the x and the second the y of the point.

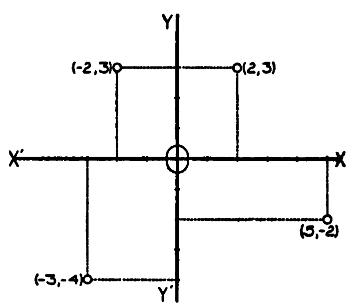


FIG. 4.

To locate the point $(-2, 3)$, we measure two units to the left along the x-axis, and three units upward parallel to the y-axis. In like manner, the points $(-3, -4)$ and $(5, -2)$ are located, as in Fig. 4.

x	y
0	-3
1	-1
2	1
3	3
-1	-5
-2	-7

40. If, in the equation $y = 2x - 3$, we give to x a series of numerical values, we obtain corresponding values of y. In the table on the left, corresponding values of x and y are written in the same horizontal line. Each set of values of x and y may be taken as the coördinates of a point in the plane, in Fig. 5.

If we give to x values between those in the table, we obtain corresponding values of y. Thus, if we give to x the value $\frac{1}{2}$, between 0 and 1, we obtain for y the value -2, between -3 and -1. To each of these sets of values likewise corresponds a point. If, then, x be made to change continuously from 0 to 1, passing through every intermediate value, y will change continuously from -3 to -1, and the points corresponding will form a continuous line from $(0, -3)$ to $(1, -1)$.

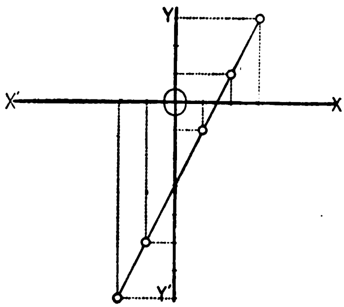


FIG. 5.

In like manner we have a continuous line connecting all the points given in Fig. 5.

41. It is proved in analytic geometry that a simple equation in two unknown numbers represents in this way a straight line. It is on this account that it is called a *linear* equation.

The figure is called the **Graph** of the equation, and the equation is said to be the **Equation of the Graph**.

42. Every solution of the equation corresponds to a point on the graph; and, conversely, the coördinates of every point on the graph, and of no other points, satisfy the equation.

We thus have, corresponding to the infinite number of solutions of an indeterminate linear equation, an infinite number of points on a straight line.

43. Since a linear equation represents a straight line, it is necessary to locate only two points to determine its position. By letting $x = 0$ in the equation of the graph, we obtain the point where the straight line crosses the y-axis, and by making $y = 0$, the point where the line crosses the x-axis.

Ex. Draw the graph of $3x + 2y = 6$.

When $x = 0$, $y = 3$; the point $(0, 3)$ is on the y-axis.

When $y = 0$, $x = 2$; the point (2, 0) is on the x-axis.

The graph is shown in Fig. 6.

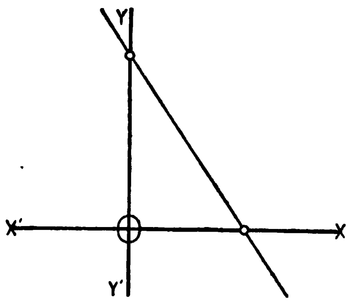


FIG. 6.

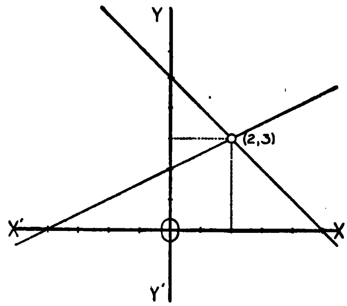


FIG. 7.

Intersection of Graphs.

44. The solution of the equations $x + y = 5$,

and $x - 2y = -4$, is 2, 3.

Since this solution satisfies both equations, it gives the coördinates of a point which is common to the graphs of these equations, as in Fig. 7. Therefore, the point of intersection of two graphs is obtained by solving their equations.

We thus see that corresponding to the one solution of two simultaneous linear equations we have the one point of intersection of two straight lines.

45. As was shown in Ch. X., a definite solution of two linear equations in two unknown numbers cannot be obtained if the equations be inconsistent or equivalent.

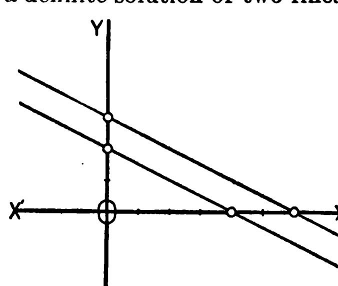


FIG. 8.

Inconsistent Equations.

46. The two equations $x + 2y = 4$, and $x + 2y = 6$, are inconsistent, and are not satisfied by any set of finite values of x and y. Drawing the graphs of these two equations, we find that they represent two parallel straight lines.

Equivalent Equations.

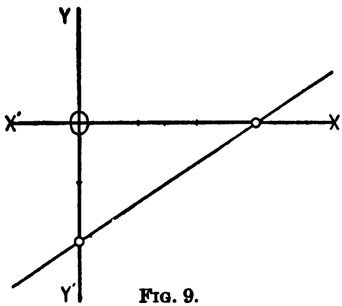


FIG. 9.

47. The two equations

$$2x - 3y = 6,$$

and $$4x - 6y = 12,$$

are equivalent, and are therefore satisfied by an indefinite number of sets of values of x and y. Drawing the graphs of these two equations, we see that they coincide, as in Fig. 9.

Graphic Representation of Roots.

48. Ex. 1. Draw the graph of the equation

$$y = x^3 - 5x^2 + 2x + 8.$$

x	$y, = f(x)$
0	8
1	6
2	0
3	-4
4	0
5	18
-1	0
-2	-24

Sets of values of x and y are given in the table on the left. The corresponding points (0, 8), (1, 6), (2, 0), etc., indicate the general outline of the curve as represented in Fig. 10.

Observe that at the points where the graph crosses the x-axis, $y = 0$; that is,

$$x^3 - 5x^2 + 2x + 8 = 0.$$

The values of x, 2, 4, -1, at these points, are therefore the roots of the above equation.

In like manner, if we place the first member of any rational integral equation, when all terms are transferred to that member, equal to y instead of 0, and draw the corresponding graph, the abscissas of the points where this graph crosses the x-axis are the roots of the equation.

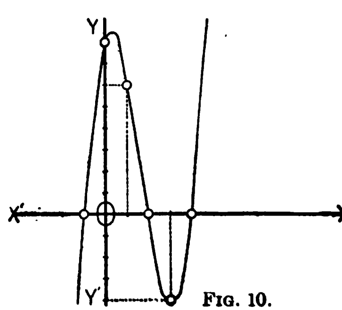


FIG. 10.

In this graphic representation the ordinate of any point

is in fact the value of the function corresponding to the value of x at the point. It will be helpful in what follows to dispense with the letter y and to denote the ordinate of a point by $f(x)$.

Ex. 2. Given the equation $x^4 - 10x^2 + 9 = 0$, draw the corresponding graph.

x	$f(x)$
0	9
1	0
2	-15
3	0
4	105
-1	0
-2	-15
-3	0
-4	105

We have $f(x) = x^4 - 10x^2 + 9.$

The graph is shown in Fig. 11.

The curve crosses the x-axis three units to the left of the origin, one unit to the left, one to the right, and three to the right. Therefore the roots of the equation are -3, -1, 1, 3.

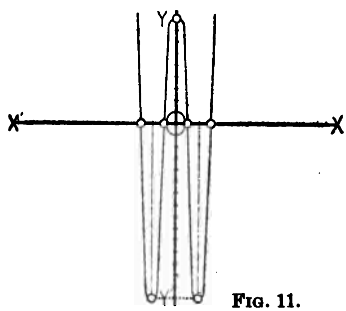


FIG. 11.

Ex. 3. Given the equation

$$x^3 - 6x^2 + 7x + 4 = 0,$$

draw the corresponding graph.

We have $f(x) = x^3 - 6x^2 + 7x + 4.$

The graph is shown in Fig. 12.

Observe that in this case two of the points where the graph crosses the x-axis, those between 2 and 3, and 0 and -1, are not accurately located. Therefore the two roots corresponding are not accurately shown by the graph. The

x	$f(x)$
0	4
1	6
2	2
3	-2
4	0
5	14
-1	-12
-2	-40

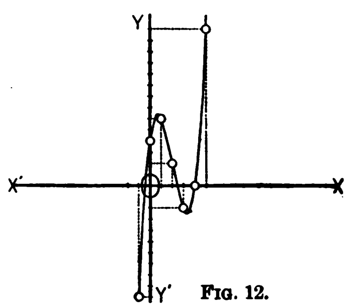


FIG. 12.

importance of graphs for our present work is not, therefore, to determine the exact values of the roots, but to make clearer some principles which are to follow.

Graphic Representation of Real and Equal, and Imaginary Roots.

49. The graphs corresponding to the three functions

$$f_1(x) = x^2 - 6x + 8, \quad (1) \qquad f_2(x) = x^2 - 6x + 9, \quad (2)$$
$$f_3(x) = x^2 - 6x + 10, \quad (3)$$

are shown in Fig. 13.

x	f_1	f_2	f_3
0	8	9	10
1	3	4	5
2	0	1	2
3	-1	0	1
4	0	1	2
5	3	4	5

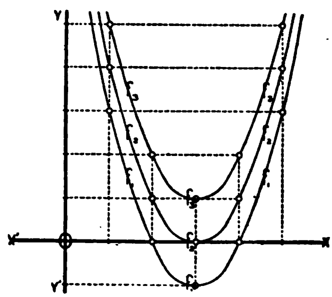


FIG. 13.

The graph corresponding to (1) cuts the x-axis in two points, two and four units to the right of the origin. Observe that 2 and 4 are the roots of the equation $x^2 - 6x + 8 = 0$.

In equation (1), when $x = 3$, $y = -1$, and the corresponding point on the graph is one unit below the x-axis.

Now observe that the second member of (2) is obtained from the second member of (1) by adding unity to the latter. Therefore the ordinates in the graph of (2) will be one greater than the corresponding ordinates in the graph of (1), as shown in the figure. This graph is tangent to the x-axis at the point 3, 0, and could have been obtained by sliding the graph of (1) vertically upward one unit. In so doing, the two points where (1) cuts the x-axis would have moved toward each other and coincided in the point where (2) touches the x-axis. The root 3 is a multiple root of the equation $x^2 - 6x + 9 = 0$.

Finally, the second member of (3) is obtained from the second member of (1) by adding two units to the latter. The

graph of (3), therefore, could have been obtained by sliding the graph of (1) vertically upward two units, as shown in the figure. This graph does not intersect the x-axis, and therefore indicates the presence of imaginary roots; in fact, the roots of the equation $x^2 - 6x + 10 = 0$ are conjugate imaginaries.

EXERCISES VIII.

1. Locate the points $(1, 3)$; $(-1, 3)$; $(2, -2)$; $(-4, -1)$; $(0, -2)$; $(3, 0)$; $(0, 4)$; $(-3, 0)$; $(0, 0)$; $(4, -2)$; $(3, 2)$; $(-5, -3)$; $(-5, 4)$.

Draw the graphs of the following equations:

2. $x = 2$. **3.** $y = -4$. **4.** $x = 0$. **5.** $y = 0$.

6. $y = 2x - 3$. **7.** $2x = y + 4$. **8.** $3x - 4y = 12$.

Draw, to one set of axes, the graphs of the equations in each of the following examples, and determine their point of intersection:

9. $\begin{cases} x + 2y = 5, \\ 3x - y = 1. \end{cases}$ **10.** $\begin{cases} 2x + 5y = -7, \\ 4x + 3y = 7. \end{cases}$

11. $\begin{cases} 3x - 2y = 6, \\ 6x - 4y = 6. \end{cases}$ **12.** $\begin{cases} x + 5y = 10, \\ 2x + 10y = 20. \end{cases}$

13. $\begin{cases} y = x + 2, \\ y = x^2 + 3x + 2. \end{cases}$ **14.** $\begin{cases} y = x - 1, \\ y = x^2 - 3x + 3. \end{cases}$

15. $\begin{cases} y = 2x - 2, \\ y = 3x^2 + 2x - 1 \end{cases}$ **16.** $\begin{cases} y = -2x + 4, \\ y = x^3 - 6x^2 + 7x + 4. \end{cases}$

Draw the graph of each of the following functions, and locate each irrational root between two consecutive integers:

17. $x^3 - 5x$. **18.** $3x^3 - 7x^2 + 4$. **19.** $2x^3 - 7x^2 - 30$.

20. $x^3 - 3x^2 - 3x + 1$. **21.** $x^3 - 5x^2 + 2x + 12$.

22. $x^3 - 7x^2 + 15x - 9$. **23.** $x^4 - 2x^3 - 7x^2 + 8x + 12$.

24. $x^4 - 9x^2 + 4x + 12$. **25.** $x^4 + 4x^3 - 9x^2 - 120$.

Greatest and Least Terms in $f(x)$.

50. *In the function*

$$f(x), = a_0x^n + \cdots + a_{n-k-1}x^{k+1} + a_{n-k}x^k + a_{n-k+1}x^{k-1} + \cdots,$$

the term $a_{n-k}x^k$ can be made to exceed numerically the sum of all the terms that follow it, by taking x large enough, and to exceed the sum of all the terms that precede it, by taking small enough.

First, let $a_{n-k}x^k = m(a_{n-k+1}x^{k-1} + \cdots + a_{n-1}x + a_n)$. (1)

Whence,

$$m = \frac{a_{n-k}x^k}{a_{n-k+1}x^{k-1} + \cdots + a_{n-1}x + a_n}, = \frac{a_{n-k}}{\frac{a_{n-k+1}}{x} + \cdots + \frac{a_{n-1}}{x^{k-1}} + \frac{a_n}{x^k}}. \quad (2)$$

By taking x large enough, we can make each term in the denominator of (2), and hence the denominator itself, as small as we please. When the denominator is made smaller than a_{n-k}, then $m > 1$, numerically. Therefore, from (1),

$$a_{n-k}x^k > a_{n-k+1}x^{k-1} + \cdots + a_{n-1}x + a_n, \text{ numerically.}$$

In like manner the second part of the principle can be proved.

51. *Since $a_0x^n \doteq \infty$, as $x \doteq \infty$, it follows from the preceding article that $f(x) \doteq \infty$, as $x \doteq \infty$.*

If we take x large enough to make the term a_0x^n exceed numerically the sum of all the terms that follow it, $f(x)$ will be positive when a_0x^n is positive, and negative when a_0x^n is negative.

Ex. 1. $\quad 3x^4 - 5x^3 + 7x^2 - 19x - 17 \doteq +\infty,$

when $\quad x \doteq \infty$, or $-\infty$.

Ex. 2. $2x^3 - 9x^2 - 6x + 5 \doteq +\infty$, when $x \doteq \infty$,

$\doteq -\infty$, when $x \doteq -\infty$.

Principle of Continuity.

52. It was assumed in Art. 40 that the curve is continuous between any two points definitely located; that is, that a point, which we may assume to be describing the curve, nowhere

makes a jump. An instance of an interruption of continuity is shown in Fig. 14. Here the point which is describing the curve on arriving at the point P, whose abscissa is a, say, jumps to the point Q. We assume that Q is a finite distance from P, while the abscissa of Q is greater than a by less than any assigned number, however small. Now, let R be a point on the curve, whose abscissa is $a+h$, h being an infinitesimal. Then, since

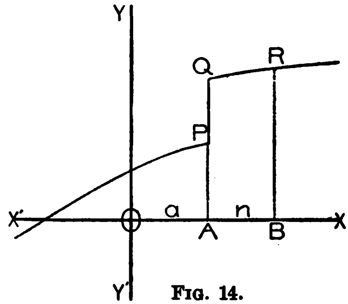


FIG. 14.

ordinates represent values of the function, we have

$$AP = f(a), \text{ and } BR = f(a+h).$$

Evidently, $f(a+h) - f(a)$ represents the change in the value of the function corresponding to the change h in the value of x.

Since $BR - AP, = f(a+h) - f(a)$, remains finite as $h \doteq 0$, corresponding to an infinitesimal change in the value of x, there is a finite change in the value of the function.

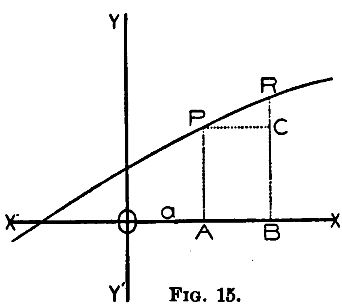


FIG. 15.

On the other hand, let the curve be continuous at a point a, as in Fig. 15. Then

$$f(a+h) - f(a), = BR - AP,$$
$$= CR.$$

As $h \doteq 0$, that is, as the point B approaches A, it is evident that CR approaches 0. That is, when the function is continuous at $x = a$, corresponding to an infinitesimal change in the value of x, there is an infinitesimal change in the value of $f(x)$.

We are thus led to the following definition of continuity:

The function, $f(x)$, is continuous at $x = a$, if

$$f(a+h) - f(a) \doteq 0, \text{ as } h \doteq 0.$$

53. *A rational integral function of x is continuous for all finite values of x.*

Let $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Then, by Art. 34,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \cdots.$$

Whence, $$f(a+h) - f(a) = h\left[f'(a) + \frac{h}{\lfloor 2}f''(a) + \frac{h^2}{\lfloor 3}f'''(a) + \cdots\right].$$

Now, as $h \doteq 0$, $\frac{h}{\lfloor 2}f''(a) \doteq 0$, $\frac{h^2}{\lfloor 3}f'''(a) \doteq 0$, and so on.

Therefore, $$f'(a) + \frac{h}{\lfloor 2}f''(a) + \frac{h^2}{\lfloor 3}f'''(a) + \cdots \doteq f'(a),$$

and $$h\left[f'(a) + \frac{h}{\lfloor 2}f''(a) + \frac{h^2}{\lfloor 3}f'''(a) + \cdots\right] \doteq 0.$$

Consequently, $$f(a+h) - f(a) \doteq 0$$

as $h \doteq 0$, and $f(x)$ is continuous at $x = a$.

But a may be any finite value of x. Therefore, $f(x)$ is continuous for all finite values of x.

54. It follows from the principle of continuity that as $f(x)$ changes from a value $f(a)$ to a value $f(b)$, it must do so by infinitesimal changes, and therefore pass through every intermediate value.

55. *If $f(a)$ and $f(b)$ have opposite signs, at least one real root of $f(x) = 0$ lies between a and b.*

By the principle of continuity, a number cannot pass from a positive value to a negative value, or *vice versa*, without passing through 0. Since $f(x)$, which is continuous in passing from $f(a)$ to $f(b)$, changes its sign, it must pass through 0. This value of x between a and b, for which $f(x)$ becomes 0, is therefore a root of $f(x) = 0$.

In connection with this principle attention is called to the graphs of the functions in Exx. 1–3, Art. 48. Thus, in Ex. 3, $f(2)$ and $f(3)$ have opposite signs, and the graph in Fig. 12 cuts the x-axis once between 2 and 3. In general, if $f(a)$ and

$f(b)$ have opposite signs, the corresponding points on the graph are on opposite sides of the x-axis. Therefore the graph must cross the x-axis at least once between a and b.

56. The following principles are derived from the preceding:

(i.) *Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term.*

For, $f(0)=a_n$; and, by Art. 51, $f(+\infty)=+\infty$, $f(-\infty)=-\infty$.

If a_n be positive, $f(0)$ and $f(-\infty)$ have opposite signs, and there is a negative root between 0 and $-\infty$.

If a_n be negative, $f(0)$ and $f(+\infty)$ have opposite signs, and there is a positive root between 0 and $+\infty$.

(ii.) *Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and one negative.*

For, $f(0)$, $=a_n$, is negative, and $f(+\infty)=+\infty$, and $f(-\infty)=+\infty$.

Since $f(0)$ and $f(+\infty)$ have opposite signs, there is a positive root between 0 and $+\infty$. For a similar reason, there is a negative root between 0 and $-\infty$.

57. The following is a more general enunciation of the principle given in Art. 55:

If $f(a)$ and $f(b)$ have opposite signs, an odd number of real roots lies between a and b.

If $f(a)$ and $f(b)$ have like signs, either no real root, or an even number of real roots, lies between a and b.

The proof is a simple extension of that given in Art. 55.

The graphs of the functions in Art. 48 throw light also on this principle. Thus, in Ex. 2, $f(-4)$ and $f(-2)$ have opposite signs, and the graph in Fig. 11 cuts the x-axis once between -4 and -2; $f(-4)$ and $f(2)$ have opposite signs, and the graph cuts the x-axis three times between -4 and 2. Again, $f(-4)$ and $f(0)$ have like signs, and the graph cuts the x-axis twice between -4 and 0; $f(-4)$ and $f(4)$ have like signs, and the graph cuts the x-axis four times between -4 and 4.

To find an equation whose roots are less than the roots of a given equation by a definite number.

58. Ex. Form the equation whose roots are 2 less than the roots of the equation $x^3-7x^2+7x+15=0$. (1)

Let y be the unknown number in the required equation.

Then $$y=x-2, \text{ or } x=y+2.$$

Substituting $y+2$ for x in equation (1), we obtain

$$(y+2)^3-7(y+2)^2+7(y+2)+15=0.$$

This is a cubic equation in y, and its simplified form can be found by performing the indicated operations, and uniting like powers of y. We thus obtain $y^3-y^2-9y+9=0$.

The following method will, however, lead to a simple way of determining the coefficients.

Let us assume the transformed equation to be

$$A_0y^3+A_1y^2+A_2y+A_3=0.$$

Substituting $x-2$ for y, we obtain

$$A_0(x-2)^3+A_1(x-2)^2+A_2(x-2)+A_3=0. \qquad (2)$$

But this is the original equation, written in a different form. Therefore any result obtained from this form will be obtained by the same process from the original form.

Dividing (2) by $x-2$, we obtain a quotient

$$A_0(x-2)^2+A_1(x-2)+A_2, \text{ and a remainder } A_3.$$

Dividing this quotient by $x-2$, we obtain a second quotient

$$A_0(x-2)+A_1, \text{ and a second remainder } A_2.$$

Dividing the second quotient by $x-2$, we obtain a quotient A_0, and a remainder A_1.

That is, *the last term in the transformed equation is the remainder obtained by dividing the given equation by $x-2$; the coefficient of y is the remainder obtained by dividing the first quotient by $x-2$; the coefficient of y^2 is the remainder obtained by dividing the second quotient by $x-2$; and so on.*

The work may be arranged compactly, thus:

$$\begin{array}{rrrr|l}
1 & -7 & 7 & 15 & (2 \\
 & 2 & -10 & -6 & \\
\hline
1 & -5 & -3 & \multicolumn{1}{|r}{9} & \\
 & 2 & -6 & \multicolumn{1}{|r}{} & \\
\cline{1-3}
1 & -3 & \multicolumn{1}{|r}{-9} & & \\
 & 2 & \multicolumn{1}{|r}{} & & \\
\cline{1-2}
1 & -1 & & &
\end{array}$$

The remainder in the first division is 9, and the coefficients of the quotient are 1, -5, -3. We therefore continue the work of the second division below, obtaining a second remainder -9. By the third division, we obtain a third remainder -1, and a final quotient 1. The remainders and the last quotient have been cut off from the rest of the work by vertical lines. These numbers from left to right are now the coefficients of the transformed equation. We therefore have

$$y^3 - y^2 - 9y + 9 = 0, \text{ as before.}$$

The method is evidently general.

59. If k be negative, the work will give an equation whose roots are k greater than the roots of the given equation.

Ex. Form the equation whose roots are 2 greater than the roots of the equation $x^4 - 6x^3 + 3x^2 + 26x - 24 = 0$.

The synthetic divisor is now -2. By the method of the preceding article, we find the required equation to be

$$y^4 - 14y^3 + 63y^2 - 90y = 0.$$

EXERCISES IX.

Find, by Art. 55 and Descartes' Rule, the first figure of each real root of the equations:

1. $x^3 - 2x^2 - 5x + 7 = 0$.
2. $x^3 - 28x + 56 = 0$.
3. $x^3 + 3x^2 - 4x - 1 = 0$.
4. $x^3 + 2x - 140 = 0$.
5. $x^3 - 12x^2 + 16x + 80 = 0$.
6. $2x^3 - 7x^2 - 30 = 0$.
7. $x^3 - 6x^2 + 9x - 3 = 0$.
8. $x^4 - 5x^3 - 4x + 19 = 0$.
9. $x^4 + 2x^3 - 16x^2 - 256 = 0$.
10. $x^4 - 6x^3 - 9x + 100 = 0$.
11. $x^4 + 4x^3 - 9x^2 - 120 = 0$.
12. $x^5 + 4x^3 - 10x^2 - 8 = 0$.
13. $x^4 - 4x^3 - 35x^2 + 38x + 126 = 0$.
14. $x^5 - 6x^4 + 6x^3 + 2x - 1 = 0$.

15–28. Find the equation whose roots are less than the roots of each equation in Exx. 1–14 by the first figure of the least real positive root.

Horner's Method of Approximation.

60. The method, which will now be given, of finding an irrational root of an equation to any required degree of accuracy is due to W. G. Horner, an English mathematician, who published it in 1819.

61. Ex. Find an irrational root of the equation

$$x^3 + x^2 - 7x - 3 = 0,$$

correct to three places of decimals.

We first find between what two integers the required root lies. The smaller of these integers is evidently the first figure of the root. The principle of Art. 55 is in many examples sufficient for this purpose, but in certain cases some other work may be necessary, as will be explained in Art. 72.

Substitute in succession 0, 1, 2, and 3, in $f(x)$.

We have $f(0) = -3$, $f(1) = -8$, $f(2) = -5$, $f(3) = 12$.

Since $f(2)$ and $f(3)$ have opposite signs, there is, by Art. 55 and Descartes' Rule, one real root between 2 and 3. The first figure of this root is therefore 2, and the root may be represented by 2 $.abcd$, wherein a, b, c, d, stand for the figures in the first, second, third, and fourth decimal places. In the vicinity of the root, the graph of $f(x)$ is as shown in Fig. 16.

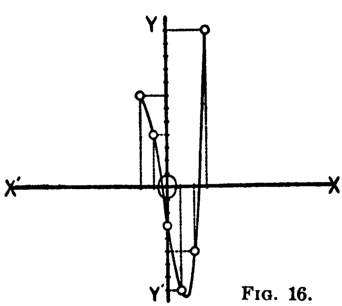


FIG. 16.

It is important to notice that if any number a little less than this root be substituted for x, the result of the substitution will be negative; and that if any number a little greater than the root be substituted for x, the result of the substitution will be positive.

Diminishing the roots of the given equation by 2, we obtain

$$\begin{array}{rrrrl} 1 & 1 & -7 & -3 & (\underline{2} \\ & 2 & 6 & -2 & \\ \hline 1 & 3 & -1 & -5 & \\ & 2 & 10 & & \\ \hline 1 & 5 & 9 & & \\ & 2 & & & \\ \hline 1 & 7 & & & \end{array}$$

$$y^3 + 7y^2 + 9y - 5 = 0.$$

The root of this equation corresponding to the required root of the given equation is evidently $.abcd$. This root lies between two consecutive tenths; that is, between 0 and .1, or .1 and .2, or .2 and .3, and so on. Since this root is a decimal, its square and its cube will be considerably smaller than its first power, and we can get a rather close approximation to the figure in the first decimal place by neglecting the terms in y^3 and y^2. We then have

$$9y - 5 = 0, \text{ whence } y = .5+.$$

Before assuming that this is the correct figure, we should, at this stage, test it by substitution. Substituting .5, we obtain 1.375. Since the result of this substitution is positive, .5 is too great.

We next try .4, and obtain $-.216$. Since this result is negative, .4 is less than the root, which therefore lies between .4 and .5, and may be represented by $.4\,bcd$. The corresponding root of the given equation is $2.4\,bcd$. In general, the figure suggested, at this stage of the work, by neglecting powers of y higher than the first will not be correct, and should be tested before continuing the work.

We now diminish the roots of the last equation by .4; that is, the roots of the original equation by 2.4.

$$\begin{array}{rrrrl} 1 & 7 & 9 & -5 & (\underline{.4} \\ & .4 & 2.96 & 4.784 & \\ \hline 1 & 7.4 & 11.96 & -\ .216 & \\ & .4 & 3.12 & & \\ \hline 1 & 7.8 & 15.08 & & \\ & .4 & & & \\ \hline 1 & 8.2 & & & \end{array}$$

The new equation is

$$z^3 + 8.2z^2 + 15.08z - .216 = 0.$$

The corresponding root of this equation is $.0bcd$; that is, it lies between 0 and .01, or .01 and .02, or .02 and .03, and so on.

As before, neglecting the terms in z^3 and z^2, we have

$$15.08\,z - .216 = 0, \text{ whence } z = .01 \text{ approximately.}$$

We may expect this value of z to give the correct figure in the second decimal place, since z^3 and z^2 are now much smaller compared with z. If .01 be too great, the result of the substitution, which is the first step in the process of forming the next equation, will be +. If it be too small, the error will be shown by the figure suggested for the next decimal place being greater than 9. Diminishing the roots of the last equation by .01, that is, the roots of the original equation by 2.41,

1	8.2	15.08	−.216	(.01
	.01	.0821	.151621	
1	8.21	15.1621	−.064379	
	.01	.0822		
1	8.22	15.2443		
	.01			
1	8.23			

The next equation is $u^3 + 8.23\,u^2 + 15.2443\,u - .064379 = 0$.

Since the last term, − .064379, is negative, we infer that .01 is not greater than the root. Neglecting the terms in u^3 and u^2, we have $15.2443\,u - .064379 = 0$, whence $u = .004+$.

Since the figure 4, suggested for the third decimal place, is not greater than 9, we infer that .01 is not too small; that is, .01 is correct. The root of the given equation obtained thus far is 2.414. Diminishing the roots of the last equation by .004,

1	8.23	15.2443	−.064379	(.004
	.004	.032936	.061108944	
1	8.234	15.277236	−.003270056	
	.004	.032952		
1	8.238	15.310188		
	.004			
1	8.242			

We now have the equation

$$v^3 + 8.242\,v^2 + 15.310188\,v - .003270056 = 0.$$

Neglecting the terms in v^3 and v^2, we obtain as the approximation to the root of this equation .0002. That is, the approximation to the root of the given equation is 2.4142.

Since the figure in the fourth decimal place is less than 5, the required root, correct to three decimal places, is 2.414.

The different steps need not be kept separate, but may be arranged compactly as follows:

1	1	− 7	− 3	(2.4142
	2	6	− 2	
1	3	− 1	− 5	
	2	10		
1	5	9		
	2			
1	7	9	− 5	
	.4	2.96	4.784	
1	7.4	11.96	− .216	
	.4	3.12		
1	7.8	15.08		
	.4			
1	8.2	15.08	− .216	
	.01	.0821	.151621	
1	8.21	15.1621	− .064379	
	.01	.0822		
1	8.22	15.2443		
	.01			
1	8.23	15.2443	− .064379	
	.004	.032936	.061108944	
1	8.234	15.277236	− .003270056	
	.004	.032952		
1	8.238	15.310188		
	.004			
1	8.242			

It is not necessary to write out each new equation in order to find the first figure of the corresponding root. It is evident that this figure is, approximately, the first figure of the quotient obtained by dividing the last term of the equation by the

coefficient of the first power of the unknown number, neglecting sign.

If the coefficient of the first power of the unknown number be 0 at any stage, the corresponding figure of the root is obtained by dividing by the coefficient of the second power of the unknown number, and taking its square root. For, the approximate equation is then of the form $my^2 - n = 0$, or $y = \sqrt{\frac{n}{m}}$.

62. The example of the preceding article illustrates the following method:

Find the integral part of the root by the method of Art. 55.

Form the equation whose roots are less than the roots of the given equation by this number.

Find the next figure of the root by dividing the last term of the transformed equation by the coefficient of the first power of the unknown number, neglecting sign, and checking the first figure of the quotient.

Form the equation whose roots are less by this number than the roots of the first transformed equation.

Take as the next figure of the root the first figure of the quotient obtained by dividing the last term of the second transformed equation by the coefficient of the first power of the unknown number; and so on.

The last term of each transformed equation must have the same sign as the last term of the given equation.

If, for any transformed equation, too great a figure be obtained by the required division, the sign of the last term of the next transformed equation will be opposite to that of the given equation.

If, for any transformed equation, too small a figure be obtained by the required division, the number suggested for the next decimal place will be greater than 9.

63. In applying Horner's Method, the decimal point can be avoided as follows, referring to the example of Art. 61.

The decimal point is about to appear in the second stage of the work, the root of the first transformed equation being *.abcd.*

Before finding the value of a, let us form the equation whose roots are 10 times the roots of this equation, as in Art. 19. The coefficients of the resulting equation are 1, 70, 900, -5000.

The corresponding root of this equation is $a.bcd$. We now have $\frac{5000}{900} = 5$. As in Art. 61, we find that the root is less than 5 and greater than 4; that is, is $4.bcd$.

Diminishing the roots by 4, we obtain a transformed equation whose corresponding root is $.bcd$.

As before, we multiply the roots of this equation by 10, and obtain as the next figure of the root, 1, $= \frac{216000}{150800}$, approximately. The work follows:

$$
\begin{array}{rrrrl}
1 & 1 & -7 & -3 & \underline{(2.4142} \\
 & 2 & 6 & -2 & \\
\hline
1 & 3 & -1 & -5 & \\
 & 2 & 10 & & \\
\hline
1 & 5 & 9 & & \\
 & 2 & & & \\
\hline
1 & 70 & 900 & -5000 & \\
 & 4 & 296 & 4784 & \\
\hline
1 & 74 & 1196 & -216 & \\
 & 4 & 312 & & \\
\hline
1 & 78 & 1508 & & \\
 & 4 & & & \\
\hline
1 & 820 & 150800 & -216000 & \\
 & 1 & 821 & 151621 & \\
\hline
1 & 821 & 151621 & -64379 & \\
 & 1 & 822 & & \\
\hline
1 & 822 & 152443 & & \\
 & 1 & & & \\
\hline
1 & 8230 & 15244300 & -64379000 & \\
 & 4 & 32936 & 61108944 & \\
\hline
1 & 8234 & 15277236 & -3270056 & \\
 & 4 & 32952 & & \\
\hline
1 & 8238 & 15310188 & & \\
 & 4 & & & \\
\hline
1 & 8242 & & &
\end{array}
$$

In general, *when the decimal point is about to appear, annex one cipher to the second coefficient (counting from the left), two to the third, and so on. Proceed as in Art.* 61, *noting that the synthetic divisor is then an integer.*

Proceed in like manner with each transformed equation.

64. A negative irrational root is obtained by first transforming the equation to one whose roots are those of the given equation with signs changed, and finding the corresponding positive root of this equation.

Roots of Numbers.

65. An approximate value of any real root of any number can be obtained by Horner's Method.

From $x = \sqrt[q]{n}$, we have $x^q - n = 0$.

The q roots of this equation are the q qth roots of n.

EXERCISES X.

1–14. Find the irrational roots, correct to three decimal places, of each of the equations in Exercises IX., Exx. 1–14.

The following equations have each two imaginary roots. Find the real rational roots; remove them and find the irrational roots of the depressed equation. Remove these roots, and solve the final depressed equation for the imaginary roots:

15. $x^4 - 11x^3 + 38x^2 - 51x + 27 = 0.$

16. $x^5 - 2x^4 - 16x^3 - 24x^2 + 144x + 320 = 0.$

17. $x^5 + 3x^4 - 7x^3 - 18x^2 - 11x - 60 = 0.$

18. $2x^6 - 8x^5 - 11x^4 - 6x^3 + 48x^2 + 38x - 15 = 0.$

Each of the following equations has two rational roots which can be expressed as decimal fractions. Find them, by Horner's Method, and solve the depressed equation:

19. $32x^3 - 80x^2 - 66x + 189 = 0.$

20. $64x^4 - 713x^2 + 732x + 247 = 0.$

21. $40x^4 - 13x^3 - 328x^2 - 13x - 368 = 0.$

Find, correct to three decimal places, by Horner's Method, the values of the following arithmetical roots:

22. $\sqrt[3]{9}$. **23.** $\sqrt[3]{2.5}$. **24.** $\sqrt[4]{8}$. **25.** $\sqrt[4]{9.2}$. **26.** $\sqrt[5]{5.7}$.

Sturm's Theorem.

66. Descartes' Rule does not give the exact number of positive and negative roots of an equation. The principle of Art. 57 does not inform us definitely whether there is one or an odd number of roots between a and b, when $f(a)$ and $f(b)$ have opposite signs; whether there is no root or an even number of roots between a and b, when $f(a)$ and $f(b)$ have the same sign. The following theorem, discovered by Sturm, a Swiss mathematician, in 1829, gives the exact number of real positive and negative roots of an equation, and also the exact number which lie in any interval. Before applying this theorem, it is assumed that all the multiple roots, if any, have been obtained and the depressed equation formed.

Let $f(x)=0$ be this depressed equation.

Then, $f(x)$ and $f'(x)$ have no common factor. In the method now to be employed, the process of finding the H. C. F. of two expressions is continued until a remainder free of x is obtained. *But the sign of each remainder is changed before it is used in the next stage as a divisor.*

Let $f_1(x)$, $f_2(x)$, etc., be the remainders, with signs changed.

Then the functions, $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $\cdots$ are called **Sturm's Functions.** We now have

$$f(x)=Q_1f'(x)+R_1, \text{ or since } R_1=-f_1(x),$$

$$f(x)=Q_1f'(x)-f_1(x); \tag{1}$$

In like manner, $$f'(x)=Q_2f_1(x)-f_2(x); \tag{2}$$

$$f_1(x)=Q_3f_2(x)-f_3(x); \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$f_k(x)=Q_{k+2}f_{k+1}(x)-f_{k+2}(x).$$

$$\cdots\cdots\cdots\cdots$$

67. The following principles are derived from the relations of Art. 66:

(i.) *Two consecutive functions cannot reduce to* 0 *for the same value of* x.

Suppose $f_2(x)$ and $f_3(x)$ reduce to 0 for the same value of x, that is, have a common factor of the form $x - r$. Then, from equation (3), $f_1(x)$ and $f_2(x)$ reduce to 0 for this value of x, and have a common factor $x - r$. In like manner, from (2), we infer that $f'(x)$ and $f_1(x)$, and finally, from (1), that $f(x)$ and $f'(x)$ reduce to 0 for this value of x; that is, have a common factor $x - r$. But this is contrary to the hypothesis that $f(x)$ and $f'(x)$ do not have a common factor.

(ii.) *Any value of x which reduces to* 0 *any function except the first gives to the two adjacent functions opposite signs.*

Let r be a value of x which reduces $f_2(x)$ to 0. Since this value of x does not reduce $f_1(x)$ or $f_3(x)$ to 0, we have

$$f_1(r) = -f_3(r).$$

The following principle is also important:

(iii.) *Just before x passes through a root r, of $f(x) = 0$, $f(x)$ and $f'(x)$ have opposite signs; and just after x has passed through a root r, of $f(x) = 0$, $f(x)$ and $f'(x)$ have the same sign.*

Let $r - h$ be a value of x just before it reaches the root r, and $r + h$ a value of x just after it has passed the root r, wherein h is a small number which approaches 0 as x *approaches* r. Then,

$$f(r+h) = f(r) + hf'(r) + \frac{h^2}{\lfloor 2}f''(r) + \cdots,$$

$$f(r-h) = f(r) - hf'(r) + \frac{h^2}{\lfloor 2}f''(r) - \cdots.$$

Since r is a root of $f(x) = 0$, $f(r) = 0$, and the above equations become

$$f(r+h) = hf'(r) + \frac{h^2}{\lfloor 2}f''(r) + \cdots, \quad (1)$$

$$f(r-h) = -hf'(r) + \frac{h^2}{\lfloor 2}f''(r) - \cdots. \quad (2)$$

By taking h small enough, the term $hf'(r)$ can be made numerically greater than the sum of all the terms which follow it. Therefore, as $h \doteq 0$, the sign of the second member will be the same as the sign of the first term, $+hf'(r)$, in equation (1), and $-hf'(r)$, in equation (2). From (1) we infer that $f(r+h)$ and $f(r)$ have like signs; that is, the sign of $f(x)$ just after x has passed the root r is the same as the sign of $f'(x)$ at the root r. But we can take the interval, h, so small that no root of $f'(x)=0$ is included between $r+h$ and r. Then, $f'(r)$ and $f'(r+h)$ will have like signs. Consequently, $f(r+h)$ and $f'(r+h)$ have like signs.

In like manner, it can be shown that $f(r-h)$ and $f'(r-h)$ have opposite signs.

68. Sturm's Theorem may now be stated as follows:

If a be substituted for x in Sturm's functions and the number of variations counted, and then b, a number greater than a, be substituted for x and the number of variations counted, the number of real roots between a and b is equal to the number of variations lost.

First, let x pass through a root r, of $f(x)=0$. Then, just before x reaches r, $f(x)$ and $f'(x)$ have opposite signs; that is, there is here one variation of sign. Just after x has passed the root r, $f(x)$ and $f'(x)$ have the same sign, and this variation is lost.

Next, let x pass through a root r, of one of the other functions, say $f_2(x)$. Then, by Art. 67 (ii.), when $x=r$, $f_1(x)$ and $f_3(x)$ have opposite signs. In passing from a value a little less than r, say $r-h$, to a value a little greater than r, say $r+h$, we can take the interval so small that no root of $f_1(x)=0$, or of $f_3(x)=0$, is included in it. Therefore $f_1(x)$ and $f_3(x)$ do not change signs in this interval. Then, just before x reaches r, and just after it has passed r, $f_1(x)$ and $f_3(x)$ have opposite signs. These may be either $+$ and $-$, or $-$ and $+$.

The sign of $f_2(x)$ will change except when r is a multiple root of $f_2(x)=0$ an even number of times.

The following arrangement of signs is now possible:

	f_1 f_2 f_3	f_1 f_2 f_3	f_1 f_2 f_3	f_1 f_2 f_3
$x = r - h$:	$+ + -$	$+ - -$	$- + +$	$- - +$
$x = r + h$:	$+ - -$	$+ + -$	$- - +$	$- + +$

Observe that the effect of the change of sign in $f_2(x)$ is to shift the variation, and not to cause it to disappear. Evidently, if $f_2(x)$ do not change its sign, there is no loss of variation.

Since a variation is lost when, and only when, x increases through a real root of $f(x) = 0$, the number of variations lost, as x passes from a to b, is the same as the number of real roots between a and b.

69. Find the number and location of the real roots of the equation $$x^4 - 4x^3 - 6x^2 + 20x + 5 = 0.$$

We have
$$f(x) = x^4 - 4x^3 - 6x^2 + 20x + 5,$$
$$f'(x) = 4x^3 - 12x^2 - 12x + 20.$$

The work of finding the other Sturm's Functions follows:

$$
\begin{array}{rl}
x^3-3x^2-3x+5) & x^4-4x^3-6x^2+20x+5\,(x-1 \\
 & \underline{x^4-3x^3-3x^2+5x} \\
 & -x^3-3x^2+15x \\
 & \underline{-x^3+3x^2+3x-5} \\
\div(-2)) & -6x^2+12x+10 \\
f_1(x)=3x^2-6x-5) & 3x^3-9x^2-9x+15\,(x-1 \\
 & \underline{3x^3-6x^2-5x} \\
 & -3x^2-4x \\
 & \underline{-3x^2+6x+5} \\
\div(-10)) & -10x+10 \\
f_2(x)=x-1) & 3x^2-6x-5\,(3x-3 \\
 & \underline{3x^2-3x} \\
 & -3x \\
 & \underline{-3x+3} \\
 & -8 \\
 & f_3(x)=8
\end{array}
$$

The first divisor is $f'(x)$ divided by 4. A positive factor may be thus introduced or suppressed, since the sign of the result is not thereby affected. Each remainder is divided by a nega-

tive factor in accordance with Art. 66, but otherwise a negative factor must not be introduced or suppressed.

We now have

$$f(x)=x^4-4x^3-6x^2+20x+5,\quad f'(x)=4x^3-12x^2-12x+20,$$

$$f_1(x)=3x^2-6x-5,\quad f_2(x)=x-1,\quad f_3(x)=8.$$

Then, by Art. 51,

x	f	f'	f_1	f_2	f_3	Var.
$-\infty$	+	−	+	−	+	4
0	+	+	−	−	+	2
$+\infty$	+	+	+	+	+	0

Since two variations are lost in passing from $-\infty$ to 0, there are two real negative roots. And since two variations are lost in passing from 0 to $+\infty$, there are two real positive roots. We next locate the positive roots and the negative roots by substituting in succession 0, 1, 2, $\cdots$, and 0, -1, -2, $\cdots$.

Observe that 1 reduces $f'(x)$ and $f_2(x)$ each to 0. This may be disregarded in counting the number of variations. Since the adjacent functions have opposite signs, the number of variations is the same as if + or − were where the 0 is. Since one variation is lost in passing from 2 to 3 one positive root lies between 2 and 3. The other positive root is found to lie between 4 and 5.

x	f	f'	f_1	f_2	f_3	Var.
0	+	+	−	−	+	2
1	+	0	−	0	+	2
2	+	−	−	+	+	2
3	−	−	+	+	+	1
4	−	+	+	+	+	1
5	+	+	+	+	+	0
0	+	+	−	−	+	2
−1	−	+	+	−	+	3
−2	−	−	+	−	+	3
−3	+	−	+	−	+	4

The negative roots lie between 0 and -1, and -2 and -3.

70. The following graphic representation of the Sturm's Functions will throw some light on how the various signs change and how the variations are lost.

Let a, b, c, d, etc., be points on the x-axis between each pair of roots of any of the Sturm's Functions, in Fig. 17.

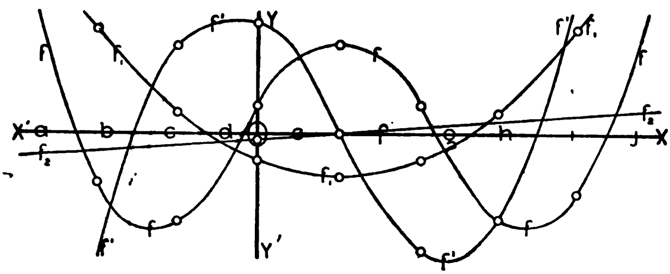


FIG. 17.

The following table shows the signs of the different functions at these points, taken from their graphs.

	f	f'	f_1	f_2	f_3	Var.
a	+	−	+	−	+	4
b	−	−	+	−	+	3
c	−	+	+	−	+	3
d	−	+	−	−	+	3
e	+	+	−	−	+	2
f	+	−	−	+	+	2
g	−	−	−	+	+	1
h	−	−	+	+	+	1
i	−	+	+	+	+	1
j	+	+	+	+	+	0

Observe, first, that a variation is always lost when x passes through a root of $f(x)$, as from a to b, from f to g, etc. Also, that as x passes through a root of any other function, the signs of the functions are so distributed that no variation is lost, but a variation is moved to the left toward $f(x)$ and $f'(x)$, as in passing from b to c through a root of $f'(x) = 0$, from g to h through a root of $f_1(x) = 0$, etc.

Observe, also, that although just after x has passed a root of $f(x)$ a permanence is established between $f(x)$ and $f'(x)$, yet before another root of $f(x)$ is reached, the distribution of signs has changed this permanence into a variation without the loss of a variation elsewhere. Thus, the permanence established at b is changed to a variation at c, which is maintained at d and lost at e. Note how the variation at e between $f_2(x)$ and $f_3(x)$ is gradually shifted to the left, to be lost at j.

71. Sturm's Theorem gives the exact number of imaginary roots, since this number is equal to the degree of the equation less the number of real roots.

Roots nearly Equal.

72. Sturm's Theorem gives the exact number and location of all the real roots of an equation, but, in application, it is very laborious.

As a rule, the method given in Art. 55 is sufficient for determining the location of the real roots. But if two or more roots lie between the same pair of consecutive integers, it is necessary to apply Sturm's Theorem. The following example will illustrate this special case.

Ex. 1. Solve the equation $x^4 - 4x^3 + x^2 + 6x + 2 = 0$.

By inspection, as in Art. 58, we find that 4 is a superior limit to the real roots, and -2 an inferior limit. By Descartes' rule of signs the equation cannot have more than two real positive roots, or more than two real negative roots.

Substituting in succession 0, 1, 2, 3, 4, -1, -2, that is, all consecutive integers within the limits to the roots, we obtain always positive results. We therefore conclude either that the roots are all imaginary, or that an even number of roots lie between two consecutive integers. To settle this question, we apply Sturm's Theorem. We have

$$f(x) = x^4 - 4x^3 + x^2 + 6x + 2, \quad f'(x) = 4x^3 - 12x^2 + 2x + 6,$$
$$f_1(x) = 5x^2 - 10x - 7, \quad f_2(x) = x - 1, \quad f_3(x) = 12.$$

x	f	f'	f_1	f_2	f_3	Var.
$-\infty$	+	−	+	−	+	4
0	+	+	−	−	+	2
$+\infty$	+	+	+	+	+	0
1	+	0	−	0	+	2
2	+	−	−	+	+	2
3	+	+	+	+	+	0
-1	+	−	+	−	+	4

We infer from the table on the left that the equation has two real positive roots and two real negative roots.

Since two variations are lost in passing from 2 to 3, the equation has two real roots between 2 and 3. For the same reason, it has two real roots between 0 and -1.

We proceed to find the two roots between 2 and 3, the integral part of each being 2.

Diminishing the roots by 2, we find the transformed equation, with its roots multiplied by 10, to be

$$y^4 + 40\,y^3 + 100\,y^2 - 6000\,y + 20000 = 0.$$

This equation has two roots between 0 and 10. Substituting successively 0, 1, 2, ..., we find that one root lies between 4 and 5, and one between 7 and 8. Hence, the required roots to the first decimal place are 2.4 and 2.7.

The roots being now separated, we proceed with each by itself, in the regular way. The two roots, to four decimal places, are found to be 2.4142 and 2.7321.

Had the two roots of the transformed equation not been separated, it would have been necessary to apply Sturm's Theorem to it, as to the original equation.

The two roots between 0 and -1 can be found in like manner, first transforming the equation into another whose roots are those of the given equation with signs changed.

EXERCISES XI.

Determine the number and location of the real roots of the following equations. Also, find each real positive root correct to three decimal places:

1. $x^3 - 9\,x - 12 = 0.$

2. $x^3 - 6\,x^2 + 8\,x - 1 = 0.$

3. $x^3 - 3\,x^2 - 4\,x + 13 = 0.$

4. $x^3 - 27\,x + 246 = 0.$

5. $x^3 + 3\,x^2 - 4\,x + 1 = 0.$

6. $x^4 - 11\,x^2 + 28\,x - 6 = 0.$

7. $x^4 - 4\,x^3 - 8\,x^2 + 40\,x - 20 = 0.$

8. $x^4 - 6\,x^3 + 11\,x^2 - 10\,x + 2 = 0.$

9. $x^4 - 8\,x^3 + 10\,x^2 + 40\,x - 63 = 0.$

10. $x^4 - 2\,x^3 - 17\,x^2 + 20\,x + 70 = 0.$

11. $2\,x^4 - 4\,x^3 - 5\,x^2 + 12\,x - 3 = 0.$

12. $5\,x^4 - 10\,x^3 - 36\,x^2 + 60\,x + 36 = 0.$

Reciprocal Equations.

73. *To transform an equation into another whose roots are the reciprocals of the roots of the given equation.*

Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0$
be the given equation.

If $y = \dfrac{1}{x}$, or $x = \dfrac{1}{y}$, then each value of y is the reciprocal of a value of x. Substituting $\dfrac{1}{y}$ for x, and multiplying by y^n,

$$a_ny^n + a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \cdots + a_2y^2 + a_1y + a_0 = 0.$$

That is, the coefficients of the transformed equation are the coefficients of the given equation written in reverse order.

Ex. The equation whose roots are the reciprocals of the roots of the equation $2x^3 - 3x^2 + 7x + 5 = 0$
is
$$5y^3 + 7y^2 - 3y + 2 = 0.$$

74. **A Reciprocal Equation** is one whose roots, in pairs, are reciprocals one of the other.

That is, if r be a root of a reciprocal equation, $\dfrac{1}{r}$ is also a root.

75. Now, if r be a root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

then, by Art. 73, $\dfrac{1}{r}$ is a root of the equation

$$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0. \tag{2}$$

Equations (1) and (2) will in general be different; but if equation (1) be a reciprocal equation, $\dfrac{1}{r}$ will be a root of it also.

That is, each root of (2) is a root of (1), and *vice versa*, and the two equations are equivalent. Hence, corresponding coefficients must be equal or proportional, and

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \cdots = \frac{a_{n-1}}{a_1} = \frac{a_n}{a_0}.$$

Whence, $\qquad a_0^2 = a_n^2,\ a_1^2 = a_{n-1}^2, \cdots,$

or $\qquad a_0 = \pm a_n,\ a_1 = \pm a_{n-1}, \cdots.$

That is, in a reciprocal equation, the coefficients of terms equally distant from the beginning and end are equal, or equal and opposite. Thus,

$$2x^3 + 3x^2 + 3x + 2 = 0, \text{ and } ax^4 - bx^3 + bx - a = 0,$$

are reciprocal equations.

Observe that an equation of even degree has one middle term. When the equidistant coefficients are opposite in sign, this term must be wanting, since it cannot be equal and opposite to itself.

Reciprocal Equation of Odd Degree.

76. Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \pm a_2x^2 \pm a_1x \pm a_0 = 0$

be a reciprocal equation of odd degree.

Grouping equidistant terms, we have

$$a_0(x^n \pm 1) + a_1x(x^{n-2} \pm 1) + a_2x^2(x^{n-4} \pm 1) + \cdots = 0,$$

wherein the upper signs go together, and also the lower. When the upper signs are taken, the equation is divisible by $x + 1$, and -1 is a root. When the lower signs are taken, the equation is divisible by $x - 1$, and 1 is a root.

When $x^n + 1$ is divided by $x + 1$, or $x^n - 1$ by $x - 1$, the first and last terms of the quotient have the same sign. These terms, multiplied by a_0, are the first and last terms of the depressed equation. In like manner, it follows that other equidistant terms in the depressed equation have the same sign.

Hence, *a reciprocal equation of an odd degree has a root* -1 *when equidistant coefficients have the same sign, and a root* $+1$ *when these coefficients have opposite signs.*

The depressed equation is a reciprocal equation of even degree, in which the equidistant coefficients have the same sign.

Reciprocal Equations of Even Degree.

77. Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots - a_2x^2 - a_1x - a_0 = 0$

be a reciprocal equation of even degree, in which equidistant coefficients have opposite signs.

Grouping terms, we have

$$a_0(x^n - 1) + a_1x(x^{n-2} - 1) + a_2x^2(x^{n-4} - 1) + \cdots = 0.$$

Since n is even, $x^n - 1$ is divisible by $x^2 - 1$, and ± 1 are roots.

Therefore, *a reciprocal equation of even degree, in which the coefficients of equidistant terms are opposite, has the roots* ± 1.

The depressed equation is a reciprocal equation of even degree in which the coefficients of equidistant terms have the same sign.

78. Standard Form of Reciprocal Equations. — It follows from the preceding articles that any reciprocal equation can be reduced to one of even degree in which the coefficients of equidistant terms have the same sign, if it be not already in this form.

This is therefore taken as the *standard form* of reciprocal equations.

Ex. Reduce the equation $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$ to the standard form.

By Art. 76, this equation has the root -1.

Dividing by $x + 1$, we obtain the depressed equation

$$x^4 + x^3 - 4x^2 + x + 1 = 0.$$

79. *A reciprocal equation of the standard form can be reduced to an equation of half its degree.*

Let $\quad a_0x^{2n} + a_1x^{2n-1} + \cdots + a_nx^n + \cdots + a_1x + a_0 = 0$

be a standard reciprocal equation, wherein a_nx^n is the middle term. Dividing by x^n, and grouping terms having equal coefficients, we obtain

$$a_0\left(x^n + \frac{1}{x^n}\right) + a_1\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + \cdots + a_n = 0.$$

Let $$x + \frac{1}{x} = z.$$

Then, $\left(x^{n-1} + \dfrac{1}{x^{n-1}}\right)\left(x + \dfrac{1}{x}\right) = x^n + \dfrac{1}{x^n} + x^{n-2} + \dfrac{1}{x^{n-2}}.$

Whence, $x^n + \dfrac{1}{x^n} = z\left(x^{n-1} + \dfrac{1}{x^{n-1}}\right) - \left(x^{n-2} + \dfrac{1}{x^{n-2}}\right).$

When $\quad n=2,\ x^2+\dfrac{1}{x^2}=z^2-2;$

$$n=3,\ x^3+\frac{1}{x^3}=z(z^2-2)-z=z^3-3z;$$

and so on.

Evidently $x^n+\dfrac{1}{x^n}$ is of the nth degree in z. Therefore, making the above substitutions, we obtain an equation of the nth degree in z.

Ex. Solve the equation

$$x^6-2x^5+2x^4-2x^3+2x^2-2x+1=0.$$

Dividing by x^3 and grouping terms,

$$\left(x^3+\frac{1}{x^3}\right)-2\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-2=0.$$

Substituting z for $x+\dfrac{1}{x}$, we have

$$z^3-3z-2(z^2-2)+2z-2=0,$$

or

$$z^3-2z^2-z+2=0.$$

The roots of this equation are found to be -1, 1, 2.

Solving the equations

$$x+\frac{1}{x}=-1,\quad x+\frac{1}{x}=1,\quad x+\frac{1}{x}=2,$$

we obtain

$$\frac{-1+i\sqrt{3}}{2},\ \frac{2}{-1+i\sqrt{3}},\ \frac{1+i\sqrt{3}}{2},\ \frac{2}{1+i\sqrt{3}},\ 1,\ 1.$$

Binomial Equations.

80. An equation of the form

$$x^n-a=0\ (1),\quad \text{or}\quad x^n+a=0 \qquad (2)$$

is called a **Binomial Equation.**

Since x^n-a, or x^n+a, and the derived function nx^{n-1} do not have a common factor, the n roots of a binomial equation are distinct. From (1) and (2), we obtain $x=\sqrt[n]{a}$, or $x=\sqrt[n]{-a}$.

Since each equation has n distinct roots, we conclude that *any positive or negative number has n distinct algebraical nth roots.*

81. As in Ch. XVIII., Art. 15, Ex. 1, the n roots of the equations

$$x^n - a = 0,\ x^n + a = 0,$$

can be obtained by multiplying the n roots of the equations

$$x^n - 1 = 0,\ x^n + 1 = 0,$$

by the arithmetical nth root of a. But the latter equations are evidently reciprocal equations and can be solved by the methods of Arts. 76–79.

Ex. Solve the equation $x^5 + 1 = 0$.

We have $\qquad (x+1)(x^4 - x^3 + x^2 - x + 1) = 0.$

Whence
$$x = -1, \tag{1}$$
and
$$x^4 - x^3 + x^2 - x + 1 = 0. \tag{2}$$

From (2), by the method of Art. 79, we obtain

$$x = \tfrac{1}{4}(1 + \sqrt{5} \pm \sqrt{2\sqrt{5} - 10}),\ x = \tfrac{1}{4}(1 - \sqrt{5} \pm \sqrt{-2\sqrt{5} - 10}).$$

EXERCISES XII.

Solve the following equations:

1. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0.$ **2.** $2x^4 + 5x^3 - 21x^2 + 5x + 2 = 0.$

3. $3x^4 - 2x^3 - 34x^2 - 2x + 3 = 0.$ **4.** $6x^4 - x^3 - 14x^2 - x + 6 = 0.$

5. $x^5 - 6x^4 + 13x^3 - 13x^2 + 6x - 1 = 0.$

6. $3x^5 - x^4 - 2x^3 - 2x^2 - x + 3 = 0.$

7. $4x^6 + 29x^5 + 55x^4 - 55x^2 - 29x - 4 = 0.$

8. $12x^6 + 16x^5 - 25x^4 - 33x^3 - 25x^2 + 16x + 12 = 0.$

9. $x^4 + 1 = 0.$ **10.** $x^4 + a^2 = 0.$ **11.** $x^5 - 1 = 0.$

12. $x^5 - 32 = 0.$ **13.** $x^6 + 1 = 0.$ **14.** $x^6 + 27 = 0$

General Solutions.

82. Hitherto the only applications of the principles developed have been to the solution of numerical equations. Nowhere have we obtained a *general solution*, that is, a solution in terms of literal coefficients, from which the roots of a particular equation could be obtained by substitution. Such a general solution

of the quadratic equation was given in Ch. XVIII., Art. 7. We will now give general algebraical solutions of the cubic and the biquadratic equations. Abel has proved that it is not possible to obtain a general algebraical solution of an equation of degree higher than the fourth.

To transform an Equation into Another in which a Particular Term is Wanting.

83. Let it be required to transform the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

into another in which a particular term is wanting.

Substituting $y + k$ for x, we have

$$a_0(y+k)^n + a_1(y+k)^{n-1} + a_2(y+k)^{n-2} + \cdots = 0,$$

or,

$$a_0y^n + (na_0k + a_1)y^{n-1} + \left[\frac{n(n-1)}{2}a_0k^2 + (n-1)a_1k + a_2\right]y^{n-2} + \cdots = 0.$$

We now equate to zero the coefficient of the particular term which is to be wanting in the transformed equation, and solve the equation thus obtained for k.

If the term in y^{n-1} is to be wanting, we let $na_0k + a_1 = 0$.

Whence, $$k = -\frac{a_1}{na_0}.$$

In particular, for the general cubic, $k = -\dfrac{a_1}{3a_0}$.

Ex. Transform the equation

$$x^3 - 6x^2 + 10x - 3 = 0$$

into another in which the term of the second degree shall be wanting.

By Art. 83, $k = 2$. We now substitute $y + 2$ for x in the given equation. Since this substitution diminishes the roots of the given equation by 2, we obtain by the synthetic method, as in Art. 58, the equation $y^3 - 2y + 1 = 0$.

Cube Roots of Unity.

84. In Ch. XVIII., Art. 15, Ex. 1, we found the three cube roots of unity to be 1, $-\frac{1}{2}(1 - i\sqrt{3})$, $-\frac{1}{2}(1 + i\sqrt{3})$.

Let $\alpha = -\frac{1}{2}(1 - i\sqrt{3})$; then $\alpha^2 = \frac{1}{4}(1 - i\sqrt{3})^2 = -\frac{1}{2}(1 + i\sqrt{3})$.

Therefore we may denote the three cube roots of unity by 1, α, α^2. Since the term of the second degree is wanting in the equation $x^3 - 1 = 0$, we have, by Art. 12 (i.),

$$1 + \alpha + \alpha^2 = 0.$$

In general, if r be one of the three cube roots of a, the three roots are r, $r\alpha$, $r\alpha^2$, wherein α is one of the two imaginary cube roots of unity.

85. By the method of the preceding article, any cubic equation can be reduced to the form

$$x^3 + px + q = 0.$$

Cardan's Solution of the Cubic.

86. To solve the equation $x^3 + px + q = 0$. — Let $x = y + z$. (1)

Then,
$$(y + z)^3 + p(y + z) + q = 0,$$
or
$$y^3 + z^3 + 3yz(y + z) + p(y + z) + q = 0,$$
or
$$y^3 + z^3 + (3yz + p)(y + z) + q = 0. \qquad (2)$$

In equations (1) and (2) we have a system of two equations in the three unknown numbers x, y, z. A third equation is therefore necessary. Let this be $3yz = -p$. (3)

Then equation (2) becomes $y^3 + z^3 + q = 0$. (4)

Substituting the value of z from (3) in (4), we have

$$y^3 - \frac{p^3}{27y^3} + q = 0, \text{ or } y^6 + qy^3 = \frac{p^3}{27}. \qquad (5)$$

From (5),
$$y^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Then, from (4),

$$z^3 = -q - y^3 = -\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Therefore,

$$y + z = \sqrt[3]{\left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} + \sqrt[3]{\left(-\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)},$$

wherein the upper signs go together, and the lower together.

Observe, however, that the value of $y+z$ is the same whether the upper signs or the lower signs be taken. Hence,

$$x = y + z = \sqrt[3]{\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{7}}\right)} + \sqrt[3]{\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)}.$$

In this value of x each cube root has three values, apparently giving nine values to x. But a cubic equation can have only three roots. This apparent contradiction is explained thus.

Each pair of cube roots, values of y and z respectively, whose sum is x, must satisfy the equation $3\,yz = -p$.

Let y_1 and z_1 be any pair of cube roots which satisfy this condition. Then, by Art. 83, the three values of y are y_1, $y_1\alpha$, $y_1\alpha^2$; and of z are z_1, $z_1\alpha$, $z_1\alpha^2$.

We have $\qquad 3\,y_1z_1 = -p,$

$$3\,(y_1\alpha)(z_1\alpha^2), = 3\,y_1z_1\alpha^3, = -p,$$

and $\qquad 3\,(y_1\alpha^2)(z_1\alpha), = 3\,y_1z_1\alpha^3, = -p;$

but $\qquad 3\,(y_1\alpha)(z_1\alpha), = 3\,y_1z_1\alpha^2, \neq -p;$

and so on.

Therefore the values of $y + z$, and hence of x, are restricted to

$$y_1 + z_1,\ \ y_1\alpha + z_1\alpha^2,\ \ y_1\alpha^2 + z_1\alpha.$$

87. Although the above solution is commonly called *Cardan's Solution*, after Cardan, who first published it in 1545, it is not due to him. The solution was discovered by Tartaglia and Ferreo, independently of each other, about 1505. Cardan obtained it from Tartaglia.

88. The chief importance of Cardan's solution is in theoretical rather than in practical applications. The more general methods, previously given, are as a rule to be preferred in solving numerical equations.

Ex. Solve the equation $\qquad x^3 - 3\,x + 2 = 0.$

We have $\qquad p = -3,\ q = 2.$

Therefore, $\qquad x = \sqrt[3]{-1} + \sqrt[3]{-1} = -2.$

The depressed equation is $\quad x^2 - 2\,x + 1$; whence $\quad x = 1, 1.$

These two roots could also have been obtained as follows:

Since $y_1 = z_1 = -1$, these roots are $-\alpha - \alpha^2, = -\alpha^2 - \alpha = 1$, by Art. 84.

When two of the roots are equal, or two imaginary, the third root can be obtained by Cardan's formula.

When, however, the three roots are real and distinct, Cardan's formula does not give a practicable algebraical solution.

General Solution of the Biquadratic.

89. One of the various solutions of the biquadratic equation follows.

If the term in x^3 be present in the given equation, it is first transformed into one in which the term of the third degree is wanting. We therefore assume the biquadratic equation in the form
$$x^4 + px^2 + qx + r = 0.$$

A factor of this equation corresponding to two real roots is of the form
$$x^2 + hx + k.$$

Also, by Arts. 65–66, a factor corresponding either to a pair of conjugate surd roots, or to a pair of conjugate imaginary roots, is of a similar form. Therefore, whatever be the nature of the required roots, we may assume

$$x^4+px^2+qx+r=(x^2+hx+k)(x^2+mx+n)$$
$$=x^4+(h+m)x^3+(k+n+hm)x^2+(hn+km)x+kn.$$

Equating coefficients of like powers of x, we have

$$h+m=0, \quad (1) \qquad k+n+hm=p, \quad (2)$$
$$hn+km=q, \quad (3) \qquad kn=r. \quad (4)$$

From (1) $$m=-h.$$

Then (2), (3), (4) become

$$k+n=h^2+p, \quad (5) \qquad h(n-k)=q, \quad (6) \qquad kn=r. \quad (7)$$

From (5) and (7),
$$n-k=\sqrt{(h^4+2\,ph^2+p^2-4\,r)}.$$

Substituting this value of $n-k$ in (6), squaring, and reducing, we obtain
$$h^6+2\,ph^4+(p^2-4\,r)h^2-q^2=0.$$

This cubic equation in h^2 has always one real positive root by Art. 56 (i.). When h is known, the values of k, m, n, are easily determined.

Ex. Solve the equation $x^4 - 9x^2 - 12x + 10 = 0$.

Here $\qquad p = -9,\ q = -12,\ r = 10.$

Hence the cubic in h^2 is

$$h^6 - 18h^4 + 41h^2 - 144 = 0.$$

A root of this equation is found to be 16. Whence $h = \pm 4$.

Let us take $h = 4$. Then $m = -4$, and from (5) and (7),

$$k + n = 7,\ kn = 10;\ \text{whence } k - n = \pm 3.$$

Taking $k - n = 3$, we obtain $k = 5$, $n = 2$.

The required factors are

$$x^2 + 4x + 5 \text{ and } x^2 - 4x + 2.$$

Taking $k - n = -3$, we obtain $k = 2$, $n = 5$.

But these values do not satisfy (6), since $4(5-2) \neq -12$, and are therefore not admissible.

From $\qquad x^2 + 4x + 5 = 0$, we obtain $x = -2 \pm \sqrt{-1}$;

and from $\qquad x^2 - 4x + 2 = 0, \qquad x = 2 \pm \sqrt{2}$.

Had we taken $h = -4$, we should have obtained $m = 4$, $k = 2$, $n = 5$. These values give the same factors as before.

EXERCISE XIII.

Solve the following equations:

1. $x^3 + 6x - 7 = 0$.
2. $x^3 - 27x + 54 = 0$.
3. $x^3 - 12x + 16 = 0$.
4. $x^3 - 9x + 28 = 0$.
5. $x^3 + 24x - 56 = 0$.
6. $x^3 - 48x - 128 = 0$.
7. $4x^3 - 3x + 1 = 0$.
8. $x^3 + 72x + 152 = 0$.
9. $x^3 + 9x^2 + 21x + 18 = 0$.
10. $x^3 - 6x^2 - 12x + 112 = 0$.
11. $3x^3 + 3x^2 - 11x + 21 = 0$.
12. $x^3 - 9x^2 + 171x - 755 = 0$.
13. $x^4 - 6x^2 - 3x + 2 = 0$.
14. $x^4 - 11x^2 + 10x + 6 = 0$.
15. $x^4 + x^2 + 36x + 52 = 0$.
16. $x^4 - 15x^2 - 42x - 40 = 0$.
17. $x^4 - 4x^3 + 11x^2 - 8x + 18 = 0$.
18. $x^4 - 2x^3 - 6x^2 + x + 2 = 0$.
19. $x^4 + 3x^3 - 4x^2 + 16x + 8 = 0$.
20. $3x^4 - 20x^3 - 30x^2 + 40x - 8 = 0$.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.			
100	00 000	043	087	130	173	217	260	303	346	389		**44**	**43**	**42**
01	432	475	518	561	604	647	689	732	775	817	1	4.4	4.3	4.2
02	860	903	945	988	*030	*072	*115	*157	*199	*242	2	8.8	8.6	8.4
03	01 284	326	368	410	452	494	536	578	620	662	3	13.2	12.9	12.6
04	703	745	787	828	870	912	953	995	*036	*078	4	17.6	17.2	16.8
05	02 119	160	202	243	284	325	366	407	449	490	5	22.0	21.5	21.0
06	531	572	612	653	694	735	776	816	857	898	6	26.4	25.8	25.2
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302	7	30.8	30.1	29.4
08	03 342	383	423	463	503	543	583	623	663	703	8	35.2	34.4	33.6
09	743	782	822	862	902	941	981	*021	*060	*100	9	39.6	38.7	37.8
110	04 139	179	218	258	297	336	376	415	454	493		**41**	**40**	**39**
11	532	571	610	650	689	727	766	805	844	883	1	4.1	4.0	3.9
12	922	961	999	*038	*077	*115	*154	*192	*231	*269	2	8.2	8.0	7.8
13	05 308	346	385	423	461	500	538	576	614	652	3	12.3	12.0	11.7
14	690	729	767	805	843	881	918	956	994	*032	4	16.4	16.0	15.6
15	06 070	108	145	183	221	258	296	333	371	408	5	20.5	20.0	19.5
16	446	483	521	558	595	633	670	707	744	781	6	24.6	24.0	23.4
17	819	856	893	930	967	*004	*041	*078	*115	*151	7	28.7	28.0	27.3
18	07 188	225	262	298	335	372	408	445	482	518	8	32.8	32.0	31.2
19	555	591	628	664	700	737	773	809	846	882	9	36.9	36.0	35.1
120	918	954	990	*027	*063	*099	*135	*171	*207	*243		**38**	**37**	**36**
21	08 279	314	350	386	422	458	493	529	565	600	1	3.8	3.7	3.6
22	636	672	707	743	778	814	849	884	920	955	2	7.6	7.4	7.2
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	3	11.4	11.1	10.8
24	09 342	377	412	447	482	517	552	587	621	656	4	15.2	14.8	14.4
25	691	726	760	795	830	864	899	934	968	*003	5	19.0	18.5	18.0
26	10 037	072	106	140	175	209	243	278	312	346	6	22.8	22.2	21.6
27	380	415	449	483	517	551	585	619	653	687	7	26.6	25.9	25.2
28	721	755	789	823	857	890	924	958	992	*025	8	30.4	29.6	28.8
29	11 059	093	126	160	193	227	261	294	327	361	9	34.2	33.3	32.4
130	394	428	461	494	528	561	594	628	661	694		**35**	**34**	**33**
31	727	760	793	826	860	893	926	959	992	*024	1	3.5	3.4	3.3
32	12 057	090	123	156	189	222	254	287	320	352	2	7.0	6.8	6.6
33	385	418	450	483	516	548	581	613	646	678	3	10.5	10.2	9.9
34	710	743	775	808	840	872	905	937	969	*001	4	14.0	13.6	13.2
35	13 033	066	098	130	162	194	226	258	290	322	5	17.5	17.0	16.5
36	354	386	418	450	481	513	545	577	609	640	6	21.0	20.4	19.8
37	672	704	735	767	799	830	862	893	925	956	7	24.5	23.8	23.1
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	8	28.0	27.2	26.4
39	14 301	333	364	395	426	457	489	520	551	582	9	31.5	30.6	29.7
140	613	644	675	706	737	768	799	829	860	891		**32**	**31**	**30**
41	922	953	983	*014	*045	*076	*106	*137	*168	*198	1	3.2	3.1	3.0
42	15 229	259	290	320	351	381	412	442	473	503	2	6.4	6.2	6.0
43	534	564	594	625	655	685	715	746	776	896	3	9.6	9.3	9.0
44	836	866	897	927	957	987	*017	*047	*077	*107	4	12.8	12.4	12.0
45	16 137	167	197	227	256	286	316	346	376	406	5	16.0	15.5	15.0
46	435	465	495	524	554	584	613	643	673	702	6	19.2	18.6	18.0
47	732	761	791	820	850	879	909	938	967	997	7	22.4	21.7	21.0
48	17 026	056	085	114	143	173	202	231	260	289	8	25.6	24.8	24.0
49	319	348	377	406	435	464	493	522	551	580	9	28.8	27.9	27.0
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.			

N.	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Pp. Pts.

250-299

N.	0	1	2	3	4	5	6	7	8	9	Pp.	Pts.
250	39 794	811	829	846	863	881	898	915	933	950		18
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123	1	1.8
52	40 140	157	175	192	209	226	243	261	278	295	2	3.6
53	312	329	346	364	381	398	415	432	449	466	3	5.4
54	483	500	518	535	552	569	586	603	620	637	4	7.2
55	654	671	688	705	722	739	756	773	790	807	5	9.0
56	824	841	858	875	892	909	926	943	960	976	6	10.8
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	7	12.6
58	41 162	179	196	212	229	246	263	280	296	313	8	14.4
59	330	347	363	380	397	414	430	447	464	481	9	16.2
260	497	514	531	547	564	581	597	614	631	647		17
61	664	681	697	714	731	747	764	780	797	814	1	1.7
62	830	847	863	880	896	913	929	946	963	979	2	3.4
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	3	5.1
64	42 160	177	193	210	226	243	259	275	292	308	4	6.8
65	325	341	357	374	390	406	423	439	455	472	5	8.5
66	488	504	521	537	553	570	586	602	619	635	6	10.2
67	651	667	684	700	716	732	749	765	781	797	7	11.9
68	813	830	846	862	878	894	911	927	943	959	8	13.6
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120	9	15.3
270	43 136	152	169	185	201	217	233	249	265	281		16
71	297	313	329	345	361	377	393	409	425	441	1	1.6
72	457	473	489	505	521	537	553	569	584	600	2.	3.2
73	616	632	648	664	680	696	712	727	743	759	3	4.8
74	775	791	807	823	838	854	870	886	902	917	4	6.4
75	933	949	965	981	996	*012	*028	*044	*059	*075	5	8.0
76	44 091	107	122	138	154	170	185	201	217	232	6	9.6
77	248	264	279	295	311	326	342	358	373	389	7	11.2
78	404	420	436	451	467	483	498	514	529	545	8	12.8
79	560	576	592	607	623	638	654	669	685	700	9	14.4
280	716	731	747	762	778	793	809	824	840	855		15
81	871	886	902	917	932	948	963	979	994	*010	1	1.5
82	45 025	040	056	071	086	102	117	133	148	163	2	3.0
83	179	194	209	225	240	255	271	286	301	317	3	4.5
84	332	347	362	378	393	408	423	439	454	469	4	6.0
85	484	500	515	530	545	561	576	591	606	621	5	7.5
86	637	652	667	682	697	712	728	743	758	773	6	9.0
87	788	803	818	834	849	864	879	894	909	924	7	10.5
88	939	954	969	984	*000	*015	*030	*045	*060	*075	8	12.0
89	46 090	105	120	135	150	165	180	195	210	225	9	13.5
290	240	255	270	285	300	315	330	345	359	374		14
91	389	404	419	434	449	464	479	494	509	523	1	1.4
92	538	553	568	583	598	613	627	642	657	672	2	2.8
93	687	702	716	731	746	761	776	790	805	820	3	4.2
94	835	850	864	879	894	909	923	938	953	967	4	5.6
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114	5	7.0
96	47 129	144	159	173	188	202	217	232	246	261	6	8.4
97	276	290	305	319	334	349	363	378	392	407	7	9.8
98	422	436	451	465	480	494	509	524	538	553	8	11.2
99	567	582	596	611	625	640	654	669	683	698	9	12.6
N.	0	1	2	3	4	5	6	7	8	9	Pp.	Pts.

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	61$\bar{5}$	629	643	657	671	686	700
07	714	728	742	756	770	78$\bar{5}$	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	84$\bar{5}$	859	872	886	900	914	927	941	95$\bar{5}$
16	969	982	996	*010	*024	*037	*051	*06$\bar{5}$	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	32$\bar{5}$	338	352	365
19	379	393	406	420	433	447	461	474	488	501
320	51$\bar{5}$	529	542	556	569	583	596	610	623	637
21	651	664	678	691	70$\bar{5}$	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 05$\bar{5}$	068	081	09$\bar{5}$	108	121	13$\bar{5}$	148	162	17$\bar{5}$
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	41$\bar{5}$	428	441
27	45$\bar{5}$	468	481	49$\bar{5}$	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
330	851	86$\bar{5}$	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*07$\bar{5}$	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	37$\bar{5}$	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	81$\bar{5}$	827	840	853	866	879
38	892	90$\bar{5}$	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	55$\bar{5}$	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	84$\bar{5}$	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	29$\bar{5}$	307	320	332	34$\bar{5}$	357	370	382	394
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
51	531	543	555	568	580	593	605	617	630	642
52	654	667	679	691	704	716	728	741	753	765
53	777	790	802	814	827	839	851	864	876	888
54	900	913	925	937	949	962	974	986	998	*011
55	55 023	035	047	060	072	084	096	108	121	133
56	145	157	169	182	194	206	218	230	242	255
57	267	279	291	303	315	328	340	352	364	376
58	388	400	413	425	437	449	461	473	485	497
59	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
61	751	763	775	787	799	811	823	835	847	859
62	871	883	895	907	919	931	943	955	967	979
63	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
64	56 110	122	134	146	158	170	182	194	205	217
65	229	241	253	265	277	289	301	312	324	336
66	348	360	372	384	396	407	419	431	443	455
67	467	478	490	502	514	526	538	549	561	573
68	585	597	608	620	632	644	656	667	679	691
69	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
71	937	949	961	972	984	996	*008	*019	*031	*043
72	57 054	066	078	089	101	113	124	136	148	159
73	171	183	194	206	217	229	241	252	264	276
74	287	299	310	322	334	345	357	368	380	392
75	403	415	426	438	449	461	473	484	496	507
76	519	530	542	553	565	576	588	600	611	623
77	634	646	657	669	680	692	703	715	726	738
78	749	761	772	784	795	807	818	830	841	852
79	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
81	58 092	104	115	127	138	149	161	172	184	195
82	206	218	229	240	252	263	274	286	297	309
83	320	331	343	354	365	377	388	399	410	422
84	433	444	456	467	478	490	501	512	524	535
85	546	557	569	580	591	602	614	625	636	647
86	659	670	681	692	704	715	726	737	749	760
87	771	782	794	805	816	827	838	850	861	872
88	883	894	906	917	928	939	950	961	973	984
89	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
91	218	229	240	251	262	273	284	295	306	318
92	329	340	351	362	373	384	395	406	417	428
93	439	450	461	472	483	494	506	517	528	539
94	550	561	572	583	594	605	616	627	638	649
95	660	671	682	693	704	715	726	737	748	759
96	770	780	791	802	813	824	835	846	857	868
97	879	890	901	912	923	934	945	956	966	977
98	988	999	*010	*021	*032	*043	*054	*065	*076	*086
99	60 097	108	119	130	141	152	163	173	184	195
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	13	12	11	10
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
400	60 206	217	228	239	249	260	271	282	293	304	
01	314	325	336	347	358	369	379	390	401	412	
02	423	433	444	455	466	477	487	498	509	520	
03	531	541	552	563	574	584	595	606	617	627	
04	638	649	660	670	681	692	703	713	724	735̄	
05	746	756	767	778	788	799	810	821	831	842	
06	853	863	874	885̄	895	906	917	927	938	949	11
07	959	970	981	991	*002	*013	*023	*034	*045̄	*055	1 1.1
08	61 066	077	087	098	109	119	130	140	151	162	2 2.2
09	172	183	194	204	215̄	225	236	247	257	268	3 3.3
410	278	289	300	310	321	331	342	352	363	374	4 4.4
11	384	395̄	405	416	426	437	448	458	469	479	5 5.5
12	490	500	511	521	532	542	553	563	574	584	6 6.6
13	595	606	616	627	637	648	658	669	679	690	7 7.7
14	700	711	721	731	742	752	763	773	784	794	8 8.8
15	805̄	815	826	836	847	857	868	878	888	899	9 9.9
16	909	920	930	941	951	962	972	982	993	*003	
17	62 014	024	034	045̄	055	066	076	086	097	107	
18	118	128	138	149	159	170	180	190	201	211	
19	221	232	242	252	263	273	284	294	304	315̄	
420	325̄	335	346	356	366	377	387	397	408	418	10
21	428	439	449	459	469	480	490	500	511	521	1 1.0
22	531	542	552	562	572	583	593	603	613	624	2 2.0
23	634	644	655̄	665̄	675	685	696	706	716	726	3 3.0
24	737	747	757	767	778	788	798	808	818	829	4 4.0
25	839	849	859	870	880	890	900	910	921	931	5 5.0
26	941	951	961	972	982	992	*002	*012	*022	*033	6 6.0
27	63 043	053	063	073	083	094	104	114	124	134	7 7.0
28	144	155̄	165̄	175̄	185̄	195	205	215	225	236	8 8.0
29	246	256	266	276	286	296	306	317	327	337	9 9.0
430	347	357	367	377	387	397	407	417	428	438	
31	448	458	468	478	488	498	508	518	528	538	
32	548	558	568	579	589	599	609	619	629	639	
33	649	659	669	679	689	699	709	719	729	739	
34	749	759	769	779	789	799	809	819	829	839	
35	849	859	869	879	889	899	909	919	929	939	9
36	949	959	969	979	988	998	*008	*018	*028	*038	1 0.9
37	64 048	058	068	078	088	098	108	118	128	137	2 1.8
38	147	157	167	177	187	197	207	217	227	237	3 2.7
39	246	256	266	276	286	296	306	316	326	335	4 3.6
440	345	355	365	375̄	385̄	395̄	404	414	424	434	5 4.5
41	444	454	464	473	483	493	503	513	523	532	6 5.4
42	542	552	562	572	582	591	601	611	621	631	7 6.3
43	640	650	660	670	680	689	699	709	719	729	8 7.2
44	738	748	758	768	777	787	797	807	816	826	9 8.1
45	836	846	856	865	875	885̄	895̄	904	914	924	
46	933	943	953	963	972	982	992	*002	*011	*021	
47	65 031	040	050	060	070	079	089	099	108	118	
48	128	137	147	157	167	176	186	196	205	215̄	
49	225̄	234	244	254	263	273	283	292	302	312	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
450	65 321	331	341	350	360	369	379	389	398	408	
51	418	427	437	447	456	466	475	485	495	504	
52	514	523	533	543	552	562	571	581	591	600	
53	610	619	629	639	648	658	667	677	686	696	
54	706	715	725	734	744	753	763	772	782	792	
55	801	811	820	830	839	849	858	868	877	887	10
56	896	906	916	925	935	944	954	963	973	982	1 | 1.0
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	2 | 2.0
58	66 087	096	106	115	124	134	143	153	162	172	3 | 3.0
59	181	191	200	210	219	229	238	247	257	266	4 | 4.0
460	276	285	295	304	314	323	332	342	351	361	5 | 5.0
61	370	380	389	398	408	417	427	436	445	455	6 | 6.0
62	464	474	483	492	502	511	521	530	539	549	7 | 7.0
63	558	567	577	586	596	605	614	624	633	642	8 | 8.0
64	652	661	671	680	689	699	708	717	727	736	9 | 9.0
65	745	755	764	773	783	792	801	811	820	829	
66	839	848	857	867	876	885	894	904	913	922	
67	932	941	950	960	969	978	987	997	*006	*015	
68	67 025	034	043	052	062	071	080	089	099	108	
69	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	9
71	302	311	321	330	339	348	357	367	376	385	1 | 0.9
72	394	403	413	422	431	440	449	459	468	477	2 | 1.8
73	486	495	504	514	523	532	541	550	560	569	3 | 2.7
74	578	587	596	605	614	624	633	642	651	660	4 | 3.6
75	669	679	688	697	706	715	724	733	742	752	5 | 4.5
76	761	770	779	788	797	806	815	825	834	843	6 | 5.4
77	852	861	870	879	888	897	906	916	925	934	7 | 6.3
78	943	952	961	970	979	988	997	*006	*015	*024	8 | 7.2
79	68 034	043	052	061	070	079	088	097	106	115	9 | 8.1
480	124	133	142	151	160	169	178	187	196	205	
81	215	224	233	242	251	260	269	278	287	296	
82	305	314	323	332	341	350	359	368	377	386	
83	395	404	413	422	431	440	449	458	467	476	
84	485	494	502	511	520	529	538	547	556	565	
85	574	583	592	601	610	619	628	637	646	655	
86	664	673	681	690	699	708	717	726	735	744	8
87	753	762	771	780	789	797	806	815	824	833	1 | 0.8
88	842	851	860	869	878	886	895	904	913	922	2 | 1.6
89	931	940	949	958	966	975	984	993	*002	*011	3 | 2.4
490	69 020	028	037	046	055	064	073	082	090	099	4 | 3.2
91	108	117	126	135	144	152	161	170	179	188	5 | 4.0
92	197	205	214	223	232	241	249	258	267	276	6 | 4.8
93	285	294	302	311	320	329	338	346	355	364	7 | 5.6
94	373	381	390	399	408	417	425	434	443	452	8 | 6.4
95	461	469	478	487	496	504	513	522	531	539	9 | 7.2
96	548	557	566	574	583	592	601	609	618	627	
97	636	644	653	662	671	679	688	697	705	714	
98	723	732	740	749	758	767	775	784	793	801	
99	810	819	827	836	845	854	862	871	880	888	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
500	69 897	906	914	923	932	940	949	958	966	975	
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
02	70 070	079	088	096	10$\bar{5}$	114	122	131	140	148	
03	157	165	174	183	191	200	209	217	226	234	
04	243	252	260	269	278	286	29$\bar{5}$	303	312	321	
05	329	338	346	35$\bar{5}$	364	372	381	389	398	406	9
06	415	424	432	441	449	458	467	475	484	492	1 0.9
07	501	509	518	526	535	544	552	561	569	578	2 1.8
08	586	59$\bar{5}$	603	612	621	629	638	646	65$\bar{5}$	663	3 2.7
09	672	680	689	697	706	714	723	731	740	749	4 3.6
510	757	766	774	783	791	800	808	817	825	834	5 4.5
11	842	851	859	868	876	88$\bar{5}$	893	902	910	919	6 5.4
12	927	935	944	952	961	969	978	986	99$\bar{5}$	*003	7 6.3
13	71 012	020	029	037	046	054	063	071	079	088	8 7.2
14	096	10$\bar{5}$	113	122	130	139	147	155	164	172	9 8.1
15	181	189	198	206	214	223	231	240	248	257	
16	26$\bar{5}$	273	282	290	299	307	315	324	332	341	
17	349	357	366	374	383	391	399	408	416	42$\bar{5}$	
18	433	441	4$\bar{5}$0	458	466	47$\bar{5}$	483	492	$\bar{5}$00	508	
19	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	8
21	684	692	700	709	717	725	734	742	750	759	1 0.8
22	767	775	784	792	800	809	817	825	834	842	2 1.6
23	850	858	867	875	883	892	900	908	917	92$\bar{5}$	3 2.4
24	933	941	9$\bar{5}$0	9$\bar{5}$8	966	97$\bar{5}$	983	991	999	*008	4 3.2
25	72 016	024	032	041	049	057	066	074	082	090	5 4.0
26	099	107	115	123	132	140	148	156	16$\bar{5}$	173	6 4.8
27	181	189	198	206	214	222	230	239	247	255	7 5.6
28	263	272	280	288	296	304	313	321	329	337	8 6.4
29	346	354	362	370	378	387	39$\bar{5}$	403	411	419	9 7.2
530	428	436	444	452	460	469	477	48$\bar{5}$	493	501	
31	509	518	526	534	542	550	558	567	57$\bar{5}$	583	
32	591	599	607	616	624	632	640	648	656	66$\bar{5}$	
33	673	681	689	697	705	713	722	730	738	746	
34	754	762	770	779	787	79$\bar{5}$	803	811	819	827	
35	835	843	852	860	868	876	884	892	900	908	
36	916	92$\bar{5}$	933	941	949	957	965	973	981	989	7
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	1 0.7
38	73 078	086	094	102	111	119	127	13$\bar{5}$	143	151	2 1.4
39	159	167	17$\bar{5}$	183	191	199	207	215	223	231	3 2.1
540	239	247	255	263	272	280	288	296	304	312	4 2.8
41	320	328	336	344	352	360	368	376	384	392	5 3.5
42	400	408	416	424	432	440	448	456	464	472	6 4.2
43	480	488	496	504	512	520	528	536	544	552	7 4.9
44	560	568	576	584	592	600	608	616	624	632	8 5.6
45	640	648	656	664	672	679	687	695	703	711	9 6.3
46	719	727	735	743	751	759	767	77$\bar{5}$	783	791	
47	799	807	81$\bar{5}$	823	830	838	846	854	862	870	
48	878	886	894	902	910	918	926	933	941	949	
49	957	965	973	981	989	997	*00$\bar{5}$	*013	*020	*028	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
550	74 036	044	052	060	068	076	084	092	099	107	
51	115	123	131	139	147	155̅	162	170	178	186	
52	194	202	210	218	225	233	241	249	257	265̅	
53	273	280	288	296	304	312	320	327	335	343	
54	351	359	367	374	382	390	398	406	414	421	
55	429	437	445̅	453	461	468	476	484	492	5̅00	
56	507	515	523	531	539	547	554	562	570	578	
57	586	593	601	609	617	624	632	640	648	656	
58	663	671	679	687	695̅	702	710	718	726	733	
59	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	85̅0	858	865	873	881	889	8
61	896	904	912	920	927	935̅	943	950	958	966	1 0.8
62	974	981	989	997	*005̅	*012	*020	*028	*035	*043	2 1.6
63	75 051	059	066	074	082	089	097	105̅	113	120	3 2.4
64	128	136	143	151	159	166	174	182	189	197	4 3.2
65	205̅	213	220	228	236	243	251	259	266	274	5 4.0
66	282	289	297	305̅	312	320	328	335	343	351	6 4.8
67	358	366	374	381	389	397	404	412	420	427	7 5.6
68	435̅	442	450	458	465	473	481	488	496	504	8 6.4
69	511	519	526	534	542	549	557	565̅	572	580	9 7.2
570	587	595	603	610	618	626	633	641	648	656	
71	664	671	679	686	694	702	709	717	724	732	
72	740	747	755̅	762	770	778	785	793	800	808	
73	815	823	831	838	846	853	861	868	876	884	
74	891	899	906	914	921	929	937	944	952	959	
75	967	974	982	989	997	*005̅	*012	*020	*027	*035̅	
76	76 042	05̅0	057	065̅	072	080	087	095̅	103	110	
77	118	125	133	140	148	155	163	170	178	185	
78	193	200	208	215	223	230	238	245	253	260	
79	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	
81	418	425	433	440	448	455̅	462	470	477	485̅	7
82	492	5̅00	507	515̅	522	530	537	545̅	552	559	1 0.7
83	567	574	582	589	597	604	612	619	626	634	2 1.4
84	641	649	656	664	671	678	686	693	701	708	3 2.1
85	716	723	730	738	745	753	760	768	775̅	782	4 2.8
86	790	797	805̅	812	819	827	834	842	849	856	5 3.5
87	864	871	879	886	893	901	908	916	923	930	6 4.2
88	938	945	953	960	967	975̅	982	989	997	*004	7 4.9
89	77 012	019	026	034	041	048	056	063	070	078	8 5.6
590	085	093	100	107	115̅	122	129	137	144	151	9 6.3
91	159	166	173	181	188	195	203	210	217	225̅	
92	232	240	247	254	262	269	276	283	291	298	
93	305	313	320	327	335̅	342	349	357	364	371	
94	379	386	393	401	408	415	422	430	437	444	
95	452	459	466	474	481	488	495	503	510	517	
96	525̅	532	539	546	554	561	568	576	583	590	
97	597	605̅	612	619	627	634	641	648	656	663	
98	670	677	685̅	692	699	706	714	721	728	735	
99	743	75̅0	757	764	772	779	786	793	801	808	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
01	887	895̄	902	909	916	924	931	938	945	952
02	960	967	974	981	988	996	*003	*010	*017	*025̄
03	78 032	039	046	053	061	068	075̄	082	089	097
04	104	111	118	125	132	140	147	154	161	168
05	176	183	190	197	204	211	219	226	233	240
06	247	254	262	269	276	283	290	297	305̄	312
07	319	326	333	340	347	355̄	362	369	376	383
08	390	398	405̄	412	419	426	433	440	447	455̄
09	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
11	604	611	618	625	633	640	647	654	661	668
12	675	682	689	696	704	711	718	725̄	732	739
13	746	753	760	767	774	781	789	796	803	810
14	817	824	831	838	845	852	859	866	873	880
15	888	895̄	902	909	916	923	930	937	944	951
16	958	965	972	979	986	993	*000	*007	*014	*021
17	79 029	036	043	050	057	064	071	078	085̄	092
18	099	106	113	120	127	134	141	148	155	162
19	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
21	309	316	323	330	337	344	351	358	365	372
22	379	386	393	400	407	414	421	428	435̄	442
23	449	456	463	470	477	484	491	498	505̄	511
24	518	525	532	539	546	553	560	567	574	581
25	588	595̄	602	609	616	623	630	637	644	650
26	657	664	671	678	685	692	699	706	713	720
27	727	734	741	748	754	761	768	775	782	789
28	796	803	810	817	824	831	837	844	851	858
29	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955̄	962	969	975	982	989	996
31	80 003	010	017	024	030	037	044	051	058	065̄
32	072	079	085	092	099	106	113	120	127	134
33	140	147	154	161	168	175̄	182	188	195	202
34	209	216	223	229	236	243	250	257	264	271
35	277	284	291	298	305̄	312	318	325	332	339
36	346	353	359	366	373	380	387	393	400	407
37	414	421	428	434	441	448	455̄	462	468	475
38	482	489	496	502	509	516	523	530	536	543
39	550	557	564	570	577	584	591	598	604	611
640	618	625̄	632	638	645	652	659	665	672	679
41	686	693	699	706	713	720	726	733	740	747
42	754	760	767	774	781	787	794	801	808	814
43	821	828	835̄	841	848	855̄	862	868	875	882
44	889	895	902	909	916	922	929	936	943	949
45	956	963	969	976	983	990	996	*003	*010	*017
46	81 023	030	037	043	050	057	064	070	077	084
47	090	097	104	111	117	124	131	137	144	151
48	158	164	171	178	184	191	198	204	211	218
49	224	231	238	245̄	251	258	265̄	271	278	285̄
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

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3	2.4	2.1	1.8
4	3.2	2.8	2.4
5	4.0	3.5	3.0
6	4.8	4.2	3.6
7	5.6	4.9	4.2
8	6.4	5.6	4.8
9	7.2	6.3	5.4

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
650	81 291	298	305	311	318	325	331	338	345	351	
51	358	365	371	378	385	391	398	405	411	418	
52	425	431	438	445	451	458	465	471	478	485	
53	491	498	505	511	518	525	531	538	544	551	
54	558	564	571	578	584	591	598	604	611	617	
55	624	631	637	644	651	657	664	671	677	684	
56	690	697	704	710	717	723	730	737	743	750	
57	757	763	770	776	783	790	796	803	809	816	
58	823	829	836	842	849	856	862	869	875	882	
59	889	895	902	908	915	921	928	935	941	948	7
660	954	961	968	974	981	987	994	*000	*007	*014	1 | 0.7
61	82 020	027	033	040	046	053	060	066	073	079	2 | 1.4
62	086	092	099	105	112	119	125	132	138	145	3 | 2.1
63	151	158	164	171	178	184	191	197	204	210	4 | 2.8
64	217	223	230	236	243	249	256	263	269	276	5 | 3.5
65	282	289	295	302	308	315	321	328	334	341	6 | 4.2
66	347	354	360	367	373	380	387	393	400	406	7 | 4.9
67	413	419	426	432	439	445	452	458	465	471	8 | 5.6
68	478	484	491	497	504	510	517	523	530	536	9 | 6.3
69	543	549	556	562	569	575	582	588	595	601	
670	607	614	620	627	633	640	646	653	659	666	
71	672	679	685	692	698	705	711	718	724	730	
72	737	743	750	756	763	769	776	782	789	795	
73	802	808	814	821	827	834	840	847	853	860	
74	866	872	879	885	892	898	905	911	918	924	
75	930	937	943	950	956	963	969	975	982	988	
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	
77	83 059	065	072	078	085	091	097	104	110	117	
78	123	129	136	142	149	155	161	168	174	181	
79	187	193	200	206	213	219	225	232	238	245	
680	251	257	264	270	276	283	289	296	302	308	
81	315	321	327	334	340	347	353	359	366	372	6
82	378	385	391	398	404	410	417	423	429	436	1 | 0.6
83	442	448	455	461	467	474	480	487	493	499	2 | 1.2
84	506	512	518	525	531	537	544	550	556	563	3 | 1.8
85	569	575	582	588	594	601	607	613	620	626	4 | 2.4
86	632	639	645	651	658	664	670	677	683	689	5 | 3.0
87	696	702	708	715	721	727	734	740	746	753	6 | 3.6
88	759	765	771	778	784	790	797	803	809	816	7 | 4.2
89	822	828	835	841	847	853	860	866	872	879	8 | 4.8
690	885	891	897	904	910	916	923	929	935	942	9 | 5.4
91	948	954	960	967	973	979	985	992	998	*004	
92	84 011	017	023	029	036	042	048	055	061	067	
93	073	080	086	092	098	105	111	117	123	130	
94	136	142	148	155	161	167	173	180	186	192	
95	198	205	211	217	223	230	236	242	248	255	
96	261	267	273	280	286	292	298	305	311	317	
97	323	330	336	342	348	354	361	367	373	379	
98	386	392	398	404	410	417	423	429	435	442	
99	448	454	460	466	473	479	485	491	497	504	
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700	84 510	516	522	528	535	541	547	553	559	566	
01	572	578	584	590	597	603	609	615	621	628	
02	634	640	646	652	658	665	671	677	683	689	
03	696	702	708	714	720	726	733	739	745	751	
04	757	763	770	776	782	788	794	800	807	813	
05	819	825	831	837	844	850	856	862	868	874	7
06	880	887	893	899	905	911	917	924	930	936	1 0.7
07	942	948	954	960	967	973	979	985	991	997	2 1.4
08	85 003	009	016	022	028	034	040	046	052	058	3 2.1
09	065	071	077	083	089	095	101	107	114	120	4 2.8
710	126	132	138	144	150	156	163	169	175	181	5 3.5
11	187	193	199	205	211	217	224	230	236	242	6 4.2
12	248	254	260	266	272	278	285	291	297	303	7 4.9
13	309	315	321	327	333	339	345	352	358	364	8 5.6
14	370	376	382	388	394	400	406	412	418	425	9 6.3
15	431	437	443	449	455	461	467	473	479	485	
16	491	497	503	509	516	522	528	534	540	546	
17	552	558	564	570	576	582	588	594	600	606	
18	612	618	625	631	637	643	649	655	661	667	
19	673	679	685	691	697	703	709	715	721	727	
720	733	739	745	751	757	763	769	775	781	788	6
21	794	800	806	812	818	824	830	836	842	848	1 0.6
22	854	860	866	872	878	884	890	896	902	908	2 1.2
23	914	920	926	932	938	944	950	956	962	968	3 1.8
24	974	980	986	992	998	*004	*010	*016	*022	*028	4 2.4
25	86 034	040	046	052	058	064	070	076	082	088	5 3.0
26	094	100	106	112	118	124	130	136	141	147	6 3.6
27	153	159	165	171	177	183	189	195	201	207	7 4.2
28	213	219	225	231	237	243	249	255	261	267	8 4.8
29	273	279	285	291	297	303	308	314	320	326	9 5.4
730	332	338	344	350	356	362	368	374	380	386	
31	392	398	404	410	415	421	427	433	439	445	
32	451	457	463	469	475	481	487	493	499	504	
33	510	516	522	528	534	540	546	552	558	564	
34	570	576	581	587	593	599	605	611	617	623	
35	629	635	641	646	652	658	664	670	676	682	
36	688	694	700	705	711	717	723	729	735	741	5
37	747	753	759	764	770	776	782	788	794	800	1 0.5
38	806	812	817	823	829	835	841	847	853	859	2 1.0
39	864	870	876	882	888	894	900	906	911	917	3 1.5
740	923	929	935	941	947	953	958	964	970	976	4 2.0
41	982	988	994	999	*005	*011	*017	*023	*029	*035	5 2.5
42	87 040	046	052	058	064	070	075	081	087	093	6 3.0
43	099	105	111	116	122	128	134	140	146	151	7 3.5
44	157	163	169	175	181	186	192	198	204	210	8 4.0
45	216	221	227	233	239	245	251	256	262	268	9 4.5
46	274	280	286	291	297	303	309	315	320	326	
47	332	338	344	349	355	361	367	373	379	384	
48	390	396	402	408	413	419	425	431	437	442	
49	448	454	460	466	471	477	483	489	495	500	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
51	564	570	576	581	587	593	599	604	610	616
52	622	628	633	639	645	651	656	662	668	674
53	679	685	691	697	703	708	714	720	726	731
54	737	743	749	754	760	766	772	777	783	789
55	795	800	806	812	818	823	829	835	841	846
56	852	858	864	869	875	881	887	892	898	904
57	910	915	921	927	933	938	944	950	955	961
58	967	973	978	984	990	996	*001	*007	*013	*018
59	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
61	138	144	150	156	161	167	173	178	184	190
62	195	201	207	213	218	224	230	235	241	247
63	252	258	264	270	275	281	287	292	298	304
64	309	315	321	326	332	338	343	349	355	360
65	366	372	377	383	389	395	400	406	412	417
66	423	429	434	440	446	451	457	463	468	474
67	480	485	491	497	502	508	513	519	525	530
68	536	542	547	553	559	564	570	576	581	587
69	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
71	705	711	717	722	728	734	739	745	750	756
72	762	767	773	779	784	790	795	801	807	812
73	818	824	829	835	840	846	852	857	863	868
74	874	880	885	891	897	902	908	913	919	925
75	930	936	941	947	953	958	964	969	975	981
76	986	992	997	*003	*009	*014	*020	*025	*031	*037
77	89 042	048	053	059	064	070	076	081	087	092
78	098	104	109	115	120	126	131	137	143	148
79	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
81	265	271	276	282	287	293	298	304	310	315
82	321	326	332	337	343	348	354	360	365	371
83	376	382	387	393	398	404	409	415	421	426
84	432	437	443	448	454	459	465	470	476	481
85	487	492	498	504	509	515	520	526	531	537
86	542	548	553	559	564	570	575	581	586	592
87	597	603	609	614	620	625	631	636	642	647
88	653	658	664	669	675	680	686	691	697	702
89	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
91	818	823	829	834	840	845	851	856	862	867
92	873	878	883	889	894	900	905	911	916	922
93	927	933	938	944	949	955	960	966	971	977
94	982	988	993	998	*004	*009	*015	*020	*026	*031
95	90 037	042	048	053	059	064	069	075	080	086
96	091	097	102	108	113	119	124	129	135	140
97	146	151	157	162	168	173	179	184	189	195
98	200	206	211	217	222	227	233	238	244	249
99	255	260	266	271	276	282	287	293	298	304
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Pp. Pts.

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6	3.6
7	4.2
8	4.8
9	5.4

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3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
01	363	369	374	380	38$\bar{5}$	390	396	401	407	412
02	417	423	428	434	439	44$\bar{5}$	4$\bar{5}$0	455	461	466
03	472	477	482	488	493	499	504	509	51$\bar{5}$	520
04	526	531	536	542	547	553	558	563	569	574
05	580	58$\bar{5}$	590	596	601	607	612	617	623	628
06	634	639	644	6$\bar{5}$0	655	660	666	671	677	682
07	687	693	698	703	709	714	720	725	730	736
08	741	747	752	757	763	768	773	779	784	789
09	79$\bar{5}$	800	806	811	816	822	827	832	838	843
810	849	854	859	86$\bar{5}$	870	875	881	886	891	897
11	902	907	913	918	924	929	934	940	94$\bar{5}$	950
12	956	961	966	972	977	982	988	993	998	*004
13	91 009	014	020	025	030	036	041	046	052	057
14	062	068	073	078	084	089	094	100	105	110
15	116	121	126	132	137	142	148	153	158	164
16	169	174	180	18$\bar{5}$	190	196	201	206	212	217
17	222	228	233	238	243	249	254	259	26$\bar{5}$	270
18	275	281	286	291	297	302	307	312	318	323
19	328	334	339	344	3$\bar{5}$0	35$\bar{5}$	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
21	434	440	44$\bar{5}$	450	455	461	466	471	477	482
22	487	492	498	503	508	514	519	524	529	53$\bar{5}$
23	540	545	551	556	561	566	572	577	582	587
24	593	598	603	609	614	619	624	630	63$\bar{5}$	640
25	645	651	656	661	666	672	677	682	687	693
26	698	703	709	714	719	724	730	73$\bar{5}$	740	745
27	751	756	761	766	772	777	782	787	793	798
28	803	808	814	819	824	829	834	840	84$\bar{5}$	850
29	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	9$\bar{5}$0	95$\bar{5}$
31	960	965	971	976	981	986	991	997	*002	*007
32	92 012	018	023	028	033	038	044	049	054	059
33	06$\bar{5}$	070	07$\bar{5}$	080	085	091	096	101	106	111
34	117	122	127	132	137	143	148	153	158	163
35	169	174	179	184	189	19$\bar{5}$	200	205	210	215
36	221	226	231	236	241	247	252	257	262	267
37	273	278	283	288	293	298	304	309	314	319
38	324	330	33$\bar{5}$	340	345	350	355	361	366	371
39	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
41	480	48$\bar{5}$	490	495	500	505	511	516	521	526
42	531	536	542	547	552	557	562	567	572	578
43	583	588	593	598	603	609	614	619	624	629
44	634	639	64$\bar{5}$	6$\bar{5}$0	65$\bar{5}$	660	665	670	675	681
45	686	691	696	701	706	711	716	722	727	732
46	737	742	747	752	758	763	768	773	778	783
47	788	793	799	804	809	814	819	824	829	834
48	840	84$\bar{5}$	8$\bar{5}$0	85$\bar{5}$	860	865	870	875	881	886
49	891	896	901	906	911	916	921	927	932	937
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

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1	0.6
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4	2.4
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6	3.6
7	4.2
8	4.8
9	6.4

	5
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2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
850	92 942	947	952	957	962	967	973	978	983	988	
51	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
52	93 044	049	054	059	064	069	075	080	085	090	
53	095	100	105	110	115	120	125	131	136	141	
54	146	151	156	161	166	171	176	181	186	192	
55	197	202	207	212	217	222	227	232	237	242	6
56	247	252	258	263	268	273	278	283	288	293	1 0.6
57	298	303	308	313	318	323	328	334	339	344	2 1.2
58	349	354	359	364	369	374	379	384	389	394	3 1.8
59	399	404	409	414	420	425	430	435	440	445	4 2.4
860	450	455	460	465	470	475	480	485	490	495	5 3.0
61	500	505	510	515	520	526	531	536	541	546	6 3.6
62	551	556	561	566	571	576	581	586	591	596	7 4.2
63	601	606	611	616	621	626	631	636	641	646	8 4.8
64	651	656	661	666	671	676	682	687	692	697	9 5.4
65	702	707	712	717	722	727	732	737	742	747	
66	752	757	762	767	772	777	782	787	792	797	
67	802	807	812	817	822	827	832	837	842	847	
68	852	857	862	867	872	877	882	887	892	897	
69	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	5
71	94 002	007	012	017	022	027	032	037	042	047	1 0.5
72	052	057	062	067	072	077	082	086	091	096	2 1.0
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74	151	156	161	166	171	176	181	186	191	196	4 2.0
75	201	206	211	216	221	226	231	236	240	245	5 2.5
76	250	255	260	265	270	275	280	285	290	295	6 3.0
77	300	305	310	315	320	325	330	335	340	345	7 3.5
78	349	354	359	364	369	374	379	384	389	394	8 4.0
79	399	404	409	414	419	424	429	433	438	443	9 4.5
880	448	453	458	463	468	473	478	483	488	493	
81	498	503	507	512	517	522	527	532	537	542	
82	547	552	557	562	567	571	576	581	586	591	
83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665	670	675	680	685	689	
85	694	699	704	709	714	719	724	729	734	738	4
86	743	748	753	758	763	768	773	778	783	787	1 0.4
87	792	797	802	807	812	817	822	827	832	836	2 0.8
88	841	846	851	856	861	866	871	876	880	885	3 1.2
89	890	895	900	905	910	915	919	924	929	934	4 1.6
890	939	944	949	954	959	963	968	973	978	983	5 2.0
91	988	993	998	*002	*007	*012	*017	*022	*027	*032	6 2.4
92	95 036	041	046	051	056	061	066	071	075	080	7 2.8
93	085	090	095	100	105	109	114	119	124	129	8 3.2
94	134	139	143	148	153	158	163	168	173	177	9 3.6
95	182	187	192	197	202	207	211	216	221	226	
96	231	236	240	245	250	255	260	265	270	274	
97	279	284	289	294	299	303	308	313	318	323	
98	328	332	337	342	347	352	357	361	366	371	
99	376	381	386	390	395	400	405	410	415	419	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
900	95 424	429	434	439	444	448	453	458	463	468	
01	472	477	482	487	492	497	501	506	511	516	
02	521	525	530	535	540	545̄	550̄	554	559	564	
03	569	574	578	583	588	593	598	602	607	612	
04	617	622	626	631	636	641	646	650	655	660	
05	665̄	670	674	679	684	689	694	698	703	708	
06	713	718	722	727	732	737	742	746	751	756	
07	761	766	770	775	780	785̄	789	794	799	804	
08	809	813	818	823	828	832	837	842	847	852	
09	856	861	866	871	875	880	885	890	895̄	899	5
910	904	909	914	918	923	928	933	938	942	947	1 0.5
11	952	957	961	966	971	976	980	985	990	995̄	2 1.0
12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	3 1.5
13	96 047	052	057	061	066	071	076	080	085	090	4 2.0
14	095̄	099	104	109	114	118	123	128	133	137	5 2.5
15	142	147	152	156	161	166	171	175	180	185̄	6 3.0
16	190	194	199	204	209	213	218	223	227	232	7 3.5
17	237	242	246	251	256	261	265	270	275̄	280	8 4.0
18	284	289	294	298	303	308	313	317	322	327	9 4.5
19	332	336	341	346	350	355	360	365̄	369	374	
920	379	384	388	393	398	402	407	412	417	421	
21	426	431	435	440	445̄	450̄	454	459	464	468	
22	473	478	483	487	492	497	501	506	511	515	
23	520	525̄	530	534	539	544	548	553	558	562	
24	567	572	577	581	586	591	595	600	605̄	609	
25	614	619	624	628	633	638	642	647	652	656	
26	661	666	670	675	680	685̄	689	694	699	703	
27	708	713	717	722	727	731	736	741	745	750	
28	755̄	759	764	769	774	778	783	788	792	797	
29	802	806	811	816	820	825̄	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
31	895̄	900	904	909	914	918	923	928	932	937	4
32	942	946	951	956	960	965̄	970	974	979	984	1 0.4
33	988	993	997	*002	*007	*011	*016	*021	*025	*030	2 0.8
34	97 035̄	039	044	049	053	058	063	067	072	077	3 1.2
35	081	086	090	095	100	104	109	114	118	123	4 1.6
36	128	132	137	142	146	151	155	160	165̄	169	5 2.0
37	174	179	183	188	192	197	202	206	211	216	6 2.4
38	220	225̄	230	234	239	243	248	253	257	262	7 2.8
39	267	271	276	280	285	290	294	299	304	308	8 3.2
940	313	317	322	327	331	336	340	345	350̄	354	9 3.6
41	359	364	368	373	377	382	387	391	396	400	
42	405	410	414	419	424	428	433	437	442	447	
43	451	456	460	465̄	470	474	479	483	488	493	
44	497	502	506	511	516	520	525̄	529	534	539	
45	543	548	552	557	562	566	571	575	580	585̄	
46	589	594	598	603	607	612	617	621	626	630	
47	635̄	640	644	649	653	658	663	667	672	676	
48	681	685	690	695̄	699	704	708	713	717	722	
49	727	731	736	740	745̄	749	754	759	763	768	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
51	818	823	827	832	836	841	845	850	855̄	859
52	864	868	873	877	882	886	891	896	900	905̄
53	909	914	918	923	928	932	937	941	946	950
54	955̄	959	964	968	973	978	982	987	991	996
55	98 000	005̄	009	014	019	023	028	032	037	041
56	046	050	055̄	059	064	068	073	078	082	087
57	091	096	100	105̄	109	114	118	123	127	132
58	137	141	146	150	155̄	159	164	168	173	177
59	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	25̄0	254	259	263	268
61	272	277	281	286	290	295̄	299	304	308	313
62	318	322	327	331	336	340	345̄	349	354	358
63	363	367	372	376	381	385	390	394	399	403
64	408	412	417	421	426	430	435̄	439	444	448
65	453	457	462	466	471	475	480	484	489	493
66	498	502	507	511	516	520	525̄	529	534	538
67	543	547	552	556	561	565	570	574	579	583
68	588	592	597	601	605	610	614	619	623	628
69	632	637	641	646	650	655̄	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
71	722	726	731	735	740	744	749	753	758	762
72	767	771	776	780	784	789	793	798	802	807
73	811	816	820	825̄	829	834	838	843	847	851
74	856	860	865̄	869	874	878	883	887	892	896
75	900	905̄	909	914	918	923	927	932	936	941
76	945̄	949	954	958	963	967	972	976	981	985
77	989	994	998	*003	*007	*012	*016	*021	*025	*029
78	99 034	038	043	047	052	056	061	065̄	069	074
79	078	083	087	092	096	100	105̄	109	114	118
980	123	127	131	136	140	145̄	149	154	158	162
81	167	171	176	180	185̄	189	193	198	202	207
82	211	216	220	224	229	233	238	242	247	251
83	255	260	264	269	273	277	282	286	291	295
84	300	304	308	313	317	322	326	330	335̄	339
85	344	348	352	357	361	366	370	374	379	383
86	388	392	396	401	405	410	414	419	423	427
87	432	436	441	445̄	449	454	458	463	467	471
88	476	480	484	489	493	498	502	506	511	515
89	520	524	528	533	537	542	546	550	555̄	559
990	564	568	572	577	581	585	590	594	599	603
91	607	612	616	621	625̄	629	634	638	642	647
92	651	656	660	664	669	673	677	682	686	691
93	695̄	699	704	708	712	717	721	726	730	734
94	739	743	747	752	756	760	765̄	769	774	778
95	782	787	791	795	800	804	808	813	817	822
96	826	830	835̄	839	843	848	852	856	861	865
97	870	874	878	883	887	891	896	900	904	909
98	913	917	922	926	930	935̄	939	944	948	952
99	957	961	965	970	974	978	983	987	991	996

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Pp. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Pp. Pts.

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Permutations

$_nP_n$ = no. of permutations of n things taken n at a time.

$6\cdot5\cdot4\cdot3\cdot2\cdot1 = \lfloor 6$ = factorial six

$_nP_n = \lfloor n$

$_nP_r = \lfloor n$ for r factors only.

$_nP_n$ (p alike) $= \dfrac{\lfloor n}{\lfloor p}$

$_nP_r$ (p alike) $= \dfrac{\lfloor r}{\lfloor p}$ for r factors

$_nP_n$ (p alike, q alike) $= \dfrac{\lfloor n}{\lfloor p \, \lfloor q}$

$_nP_n = \lfloor n-1$

bracelet $\frac{1}{2}\, {_nP_n} = \dfrac{\lfloor n-1}{2}$

A determinant is the symbolic arrangement in square form of the difference of two products

An equation of nth degree has n and only n roots

Assuming that every equation has a root let r_1 be a root of $f(x) = 0$.

Then $f(x) = (x - r_1) Q_1$ wherein Q_1 is of the $(n-1)$ degree

Assuming that $Q_1 = 0$ has a root r_2 then $Q_1 = (x - r_2) Q_2$, wherein Q_2 is of the $(n-2)$ degree, etc.

Continuing in this way $(n-1)$ times we obtain $(n-1)$ roots and are left with a quotient $Q_{n-1} = 0$, which is an equation of the first degree

Since at every step the equation one degree.

$\therefore$ this last quotient Q_{n-1} is one more root making n roots in all.

Let this last root be r_n, we now have

$F(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$

Since the substitution of any one of the n numbers as $(r_1, r_2 \cdots r_n)$ for x will reduce of $F(x)$ to 0. the equation $F(x) = 0$ has these n roots, and since the substitution of any number other than these will not reduce $F(x)$ to zero, then no other number can be a root of $f(x) = 0$.

$f(x) = 0$ has n and only n roots.

$f(x)$

$$x^2 - 6x + 5 = 0$$

$$(x-1)(x-5)$$

$x = 1 \quad x = 5$

$$1 - 6 + 5 = 0 \qquad 25 - 30 + 5 = 0$$

$0 = 0$ ok $\qquad 0 = 0$ ok

Proving the rule for decreasing the roots of a given equation by a definite number.

1. Given equation

(1) $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} \cdots a_{n-1} x + a_n = 0$

2. Decreasing roots by r and denoting new root by y $y = x - r$, or $x = y + r$ for x

Subst. in original equation,

(2) $a_0(y+r)^n + a_1(y+r)^{n-1} + a_2(y+r)^{n-2} \cdots a_{n-1}(y+r) + a_n = 0$

Performing indicated operations and combining like powers of y, the equation will assume form of

(3) $b_0 y^n + b_1 y^{n-1} + b_2 y^{n-2} \cdots b_{n-1} y + b_n = 0$

Substitute $(x-r)$ for y:

(4) $b_0(x-r)^n + b_1(x-r)^{n-1} + b_2(x-r)^{n-2} \cdots b_{n-1}(x-r) + b_n = 0$

Dividing (4) by $(x-r)$, we have

(5) $b_0(x-r)^{n-1} + p_1(x-r)^{n-2} + p_2(x-r)^{n-3} \cdots c_{n-1}$ with remainder of b_n.

Dividing (5) by $(x-r)$

(6) $q_0(x-r)^{n-2} + q_1(x-r)^{n-3} + q_2(x-r)^{n-4} \cdots$ etc. with R. of q_{n-1}

Continuing this process we obtain as successive remainders the coefficients of (3) from right to left which form the transformed equation.

$$\log_8 57 = \frac{\log_{10} 57}{\log_{10} 8}$$

N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
850	92 942	947	952	957	962	967	973	978	983	988	
51	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
52	93 044	049	054	059	064	069	075	080	085	090	
53	095	100	105	110	115	120	125	131	136	141	
54	146	151	156	161	166	171	176	181	186	192	
55	197	202	207	212	217	222	227	232	237	242	6
56	247	252	258	263	268	273	278	283	288	293	1 0.6
57	298	303	308	313	318	323	328	334	339	344	2 1.2
58	349	354	359	364	369	374	379	384	389	394	3 1.8
59	399	404	409	414	420	425	430	435	440	445	4 2.4
860	450	455	460	465	470	475	480	485	490	495	5 3.0
61	500	505	510	515	520	526	531	536	541	546	6 3.6
62	551	556	561	566	571	576	581	586	591	596	7 4.2
63	601	606	611	616	621	626	631	636	641	646	8 4.8
64	651	656	661	666	671	676	682	687	692	697	9 5.4
65	702	707	712	717	722	727	732	737	742	747	
66	752	757	762	767	772	777	782	787	792	797	
67	802	807	812	817	822	827	832	837	842	847	
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870	952	957	962	967	972	977	982	987	992	997	5
71	94 002	007	012	017	022	027	032	037	042	047	1 0.5
72	052	057	062	067	072	077	082	086	091	096	2 1.0
73	101	106	111	116	121	126	131	136	141	146	3 1.5
74	151	156	161	166	171	176	181	186	191	196	4 2.0
75	201	206	211	216	221	226	231	236	240	245	5 2.5
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77	300	305	310	315	320	325	330	335	340	345	7 3.5
78	349	354	359	364	369	374	379	384	389	394	8 4.0
79	399	404	409	414	419	424	429	433	438	443	9 4.5
880	448	453	458	463	468	473	478	483	488	493	
81	498	503	507	512	517	522	527	532	537	542	
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83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665	670	675	680	685	689	
85	694	699	704	709	714	719	724	729	734	738	4
86	743	748	753	758	763	768	773	778	783	787	1 0.4
87	792	797	802	807	812	817	822	827	832	836	2 0.8
88	841	846	851	856	861	866	871	876	880	885	3 1.2
89	890	895	900	905	910	915	919	924	929	934	4 1.6
890	939	944	949	954	959	963	968	973	978	983	5 2.0
91	988	993	998	*002	*007	*012	*017	*022	*027	*032	6 2.4
92	95 036	041	046	051	056	061	066	071	075	080	7 2.8
93	085	090	095	100	105	109	114	119	124	129	8 3.2
94	134	139	143	148	153	158	163	168	173	177	9 3.6
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99	376	381	386	390	395	400	405	410	415	419	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

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02	521	525	530	535	540	545	550	554	559	564	
03	569	574	578	583	588	593	598	602	607	612	
04	617	622	626	631	636	641	646	650	655	660	
05	665	670	674	679	684	689	694	698	703	708	
06	713	718	722	727	732	737	742	746	751	756	
07	761	766	770	775	780	785	789	794	799	804	
08	809	813	818	823	828	832	837	842	847	852	
09	856	861	866	871	875	880	885	890	895	899	5
910	904	909	914	918	923	928	933	938	942	947	1 0.5
11	952	957	961	966	971	976	980	985	990	995	2 1.0
12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	3 1.5
13	96 047	052	057	061	066	071	076	080	085	090	4 2.0
14	095	099	104	109	114	118	123	128	133	137	5 2.5
15	142	147	152	156	161	166	171	175	180	185	6 3.0
16	190	194	199	204	209	213	218	223	227	232	7 3.5
17	237	242	246	251	256	261	265	270	275	280	8 4.0
18	284	289	294	298	303	308	313	317	322	327	9 4.5
19	332	336	341	346	350	355	360	365	369	374	
920	379	384	388	393	398	402	407	412	417	421	
21	426	431	435	440	445	450	454	459	464	468	
22	473	478	483	487	492	497	501	506	511	515	
23	520	525	530	534	539	544	548	553	558	562	
24	567	572	577	581	586	591	595	600	605	609	
25	614	619	624	628	633	638	642	647	652	656	
26	661	666	670	675	680	685	689	694	699	703	
27	708	713	717	722	727	731	736	741	745	750	
28	755	759	764	769	774	778	783	788	792	797	
29	802	806	811	816	820	825	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
31	895	900	904	909	914	918	923	928	932	937	4
32	942	946	951	956	960	965	970	974	979	984	1 0.4
33	988	993	997	*002	*007	*011	*016	*021	*025	*030	2 0.8
34	97 035	039	044	049	053	058	063	067	072	077	3 1.2
35	081	086	090	095	100	104	109	114	118	123	4 1.6
36	128	132	137	142	146	151	155	160	165	169	5 2.0
37	174	179	183	188	192	197	202	206	211	216	6 2.4
38	220	225	230	234	239	243	248	253	257	262	7 2.8
39	267	271	276	280	285	290	294	299	304	308	8 3.2
940	313	317	322	327	331	336	340	345	350	354	9 3.6
41	359	364	368	373	377	382	387	391	396	400	
42	405	410	414	419	424	428	433	437	442	447	
43	451	456	460	465	470	474	479	483	488	493	
44	497	502	506	511	516	520	525	529	534	539	
45	543	548	552	557	562	566	571	575	580	585	
46	589	594	598	603	607	612	617	621	626	630	
47	635	640	644	649	653	658	663	667	672	676	
48	681	685	690	695	699	704	708	713	717	722	
49	727	731	736	740	745	749	754	759	763	768	
N.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.

950-999

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
51	818	823	827	832	836	841	845	850	855̄	859
52	864	868	873	877	882	886	891	896	900	905̄
53	909	914	918	923	928	932	937	941	946	950
54	955̄	959	964	968	973	978	982	987	991	996
55	98 000	005̄	009	014	019	023	028	032	037	041
56	046	050	055̄	059	064	068	073	078	082	087
57	091	096	100	105̄	109	114	118	123	127	132
58	137	141	146	150	155̄	159	164	168	173	177
59	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	25̄0	254	259	263	268
61	272	277	281	286	290	295̄	299	304	308	313
62	318	322	327	331	336	340	345̄	349	354	358
63	363	367	372	376	381	385	390	394	399	403
64	408	412	417	421	426	430	435̄	439	444	448
65	453	457	462	466	471	475	480	484	489	493
66	498	502	507	511	516	520	525̄	529	534	538
67	543	547	552	556	561	565	570	574	579	583
68	588	592	597	601	605	610	614	619	623	628
69	632	637	641	646	650	655̄	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
71	722	726	731	735	740	744	749	753	758	762
72	767	771	776	780	784	789	793	798	802	807
73	811	816	820	825̄	829	834	838	843	847	851
74	856	860	865̄	869	874	878	883	887	892	896
75	900	905̄	909	914	918	923	927	932	936	941
76	945̄	949	954	958	963	967	972	976	981	985
77	989	994	998	*003	*007	*012	*016	*021	*025	*029
78	99 034	038	043	047	052	056	061	065̄	069	074
79	078	083	087	092	096	100	105̄	109	114	118
980	123	127	131	136	140	145̄	149	154	158	162
81	167	171	176	180	185̄	189	193	198	202	207
82	211	216	220	224	229	233	238	242	247	251
83	255	260	264	269	273	277	282	286	291	295
84	300	304	308	313	317	322	326	330	335̄	339
85	344	348	352	357	361	366	370	374	379	383
86	388	392	396	401	405	410	414	419	423	427
87	432	436	441	445̄	449	454	458	463	467	471
88	476	480	484	489	493	498	502	506	511	515
89	520	524	528	533	537	542	546	550	555̄	559
990	564	568	572	577	581	585	590	594	599	603
91	607	612	616	621	625̄	629	634	638	642	647
92	651	656	660	664	669	673	677	682	686	691
93	695̄	699	704	708	712	717	721	726	730	734
94	739	743	747	752	756	760	765̄	769	774	778
95	782	787	791	795	800	804	808	813	817	822
96	826	830	835̄	839	843	848	852	856	861	865
97	870	874	878	883	887	891	896	900	904	909
98	913	917	922	926	930	935̄	939	944	948	952
99	957	961	965	970	974	978	983	987	991	996
N.	0	1	2	3	4	5	6	7	8	9

Pp. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Pp. Pts.

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Permutations

${}_nP_n$ = no. of permutations of n things taken n at a time.

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \lfloor\underline{6}$ = factorial six

${}_nP_n = \lfloor\underline{n}$

${}_nP_r = \lfloor\underline{n}$ for r factors only.

${}_nP_n$ (p alike) $= \dfrac{\lfloor\underline{n}}{\lfloor\underline{p}}$

${}_nP_r$ (p alike) $= \dfrac{\lfloor\underline{n}}{\lfloor\underline{p}}$ for r factors

${}_nP_n$ (p alike, q alike) $= \dfrac{\lfloor\underline{n}}{\lfloor\underline{p}\ \lfloor\underline{q}}$

${}_nP_n = \lfloor\underline{n-1}$

bracelet $\frac{1}{2}\ {}_nP_n = \dfrac{\lfloor\underline{n-1}}{2}$

A determinant is the symbolic arrangement in square form of the difference of two products

An equation of nth degree has n and only n roots

Assuming that every equation has a root let r_1 be a root of $f(x) = 0$.

Then $f(x) = (x - r_1) Q_1$ wherein Q_1 is of the $(n-1)$ degree

Assuming that $Q_1 = 0$ has a root r_2 then $Q_1 = (x - r_2) Q_2$, wherein Q_2 is of the $(n-2)$ degree, etc.

Continuing in this way $(n-1)$ times we obtain $(n-1)$ roots and are left with a quotient $Q_{n-1} = 0$ which is an equation of the first degree Since at every step the equation one degree.

$\therefore$ this last quotient Q_{n-1} is one more root making n roots in all.

Let this last root be r_n, we now have

$$F(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$$

Since the substitution of any one of the n numbers $a (r_1, r_2 \ldots r_n)$ for x will reduce f $F(x)$ to 0. the equation $F(x) = 0$ has these n roots, and since the substitution of any number other than these will not reduce $F(x)$ to zero, then no other number can be a root of $f(x) = 0$.

$f(x) = 0$ has $\underline{n}$ and only $\underline{n}$ roots.

$f(x)$ ... $(x-...)$ ...

... $f(...)$

$x^2-6x+5=0$

$(x-1)(x-5)$

$x=1 \quad x=5$

$1-6+5=0 \qquad 25-30+5=0$

$0=0$ ch $\qquad 0=0$ ch

Proving the rule for decreasing roots of a given equation by a definite number.

1. Given equation

$$(1)\quad a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} \cdots a_{n-1} x + a_n = 0$$

2. Decreasing roots by r and denoting new root by y $\quad y = x - r$ or $x = y + r$ for x

Subst, in original equation,

$$(2)\quad a_0(y+r)^n + a_1(y+r)^{n-1} + a_2(y+r)^{n-2} \cdots a_{n-1}(y+r) + a_n = 0$$

Performing indicated operations and combining like powers of y; the equation will assume form of

$$(3)\quad b_0 y^n + b_1 y^{n-1} + b_2 y^{n-2} \cdots b_{n-1} y + b_n = 0$$

Substitute $(x-r)$ for y:

$$(4)\quad b_0(x-r)^n + b_1(x-r)^{n-1} + b_2(x-r)^{n-2} \cdots b_{n-1}(x-r) + b_n = 0$$

Dividing (4) by $(x-r)$, we have

$$(5)\quad b_0(x-r)^{n-1} + b_1(x-r)^{n-2} + b_2(x-r)^{n-3} \cdots b_{n-1}$$ with remainder of b_n.

Dividing (5) by $(x-r)$

$$(6)\quad q_0(x-r)^{n-2} + q_1(x-r)^{n-3} + q_2(x-r)^{n-4} \cdots$$ etc. with R. of q_{n-1}

Continuing this process we obtain as successive remainders the coefficients of (3) from right to left which form the transformed equation.

$$\log_{8} 57 = \frac{\log_{10} 57}{\log_{10} 8}$$

The ordinate of any point on the graph of any equation in Y = f(x) is equal to the value of the equation when the corresponding abscissa of that point ~~is~~ is substituted in place of X in given equation.

538

20,12

$r_1 + r_2 + r_3^1$

~~$r_1 r_2$~~

$r_1 r_2 + r_1 r_3 + r_2 + \frac{3}{8}$

$r_1 r_2 r_3$ =

Plane Geom

16, 17,

1 ~~70 - 114~~

p 171 - 9 - 10

7, 8, 9, 10

y avoir // Il peut y avoir

Il doit y avoir // Il devrait //

Il n'y a pas moyen de (no possibility)

Il y a (the matter is that - -)

Bonne qualité (